Triangular Norms, Triangular Conorms, and Some Related Concepts

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Abstract

Mathematically considered, a Triangular Norm is a kind of binary operation frequently used in the context of Probabilistic Metric Spaces, but also in other very interesting fields, as may be Fuzzy Logic, or in general, in Multi-Valued Logic (MVL). The T-conorm, or S-norm, is a dual concept. Both ideas allow us to generalize the intersection and the union in a Lattice, or disjunction and conjunction in Logic. Also may be very interesting to introduce a special class of real monotone operations. We refer to the so-called Copulas, very useful in many fields. So, we offer now a comprehensive analysis of all these aggregation operators.

Keywords: mathematical analysis, measure theory, fuzzy measures, aggregation operators

1. Introducing Fuzzy Relations

The composition of fuzzy relations is defined by the so called *"max-min product",* introduced by two fuzzy relations acting on subsequent three universes of discourse, U_1, U_2, U_3,

$$R_1\,(U_1, U_2) \bullet R_2\,(U_2, U_3) = R_3\,(U_1, U_2)$$

where

$$R_3\,(U_1,U_3) = \{(x, z) : \mu_{R1 \bullet R2}\,(x, z)\}$$

being

$$\max\,\{\forall y \in U_2 : \min\,(\mu_{R1}(x, y), \mu_{R2}(y, z))\}$$

As a particular case of the previous definition for the composition of fuzzy relations, we can introduce the composition of a fuzzy set and a fuzzy relation [6, 10].

The usual properties of the classical relations can be translated to fuzzy relations [2, 3, 5], but modified in the following sense

R is *Reflexive,* if R (x, x) = 1, for any x.

Each element would be totally related with itself, when R is reflexive.

R is *Symmetric,* if R (x, y) = R (y, x), for any (x, y).

Therefore, the principal diagonal, Δ, acts as a mirror, in the associated matrix.

R is *Transitive,* not in the usual way for relations or associated matrices, but when the following holds

$$R(x, z) \geq \max\,(\min\,\{R\,(x, y), R\,(y, z)\}), \text{ for any } (x, y)$$

All these mathematical methods can be very interesting in Fuzzy Logic and also in many branches of Artificial Intelligence.

2. Connectives and Fuzzy Sets

We may define the classical operations among crisp, or classical sets, generalizing to fuzzy versions [1, 2, 8, 9]. So, they may be characterized by its membership functions.

We have, for the union of fuzzy sets, defined by

$$\text{Max } \{\mu_F(x), \mu_G(x)\}$$

And the intersection of fuzzy sets, by

$$\text{Min } \{\mu_F(x), \mu_G(x)\}$$

And also the complement of a fuzzy set, F, by

$$\mu_{c(F)}(x) = 1 - \mu_F(x)$$

The strict inclusion among two fuzzy sets may be introduced by

$$\mu_F(x) < \mu_G(x)$$

And in its more general (non strict) version,

$$\mu_F(x) \leq \mu_G(x)$$

The difference among two fuzzy sets, F and G, by

$$\mu_{F-G}(x) = \min \{\mu_F(x), \mu_{c(G)}(x)\}$$

Turning to the first mentioned definitions, both proofs can be expressed easily, by counter-examples, in an algebraic or geometric way.

3. Introduction to T-norms

A *Triangular Norm,* abridgedly expressed by *t-norm* or *T-norm,* will be a binary operation which appears in the framework of probabilistic metric spaces, but also in MVL (Multi-Valued Logic), and more specifically in Fuzzy Logic [1, 4, 7].

A *T-norm* is a function, *T,* given by

$$T: [0, 1]^2 \rightarrow [0, 1]$$

Satisfying the *properties:*

1) *Monotonicity:* if $a \leq c$ and $b \leq d$, then $T(a, b) \leq T(c, d)$
2) *Commutativity:* $T(a, b) = T(b, a)$
3) *Associativity:* $T(a, T(b, c)) = T(T(a, b), c)$
4) The number *1* acts as an *identity element,* i.e.

$$T(a, 1) = a$$

This requirement is related with that such number, 1, must correspond to the interpretation as true, being dually 0 as false.

The inner meaning of "continuity" expresses that very small changes in truth values should not macroscopically affect such values for their conjunction.

Frequently, T-norms are used to construct the intersection of fuzzy sets. But also are used as a basis for aggregation operators.

Other use of T-norms will be in Probabilistic Metric Spaces, generalizing triangle inequality of ordinary metric spaces. For this reason, it is called T-norm.

Finally, we can conclude that a T-norm generalizes the conjunction in Logic, and also the intersection in a lattice.

4. Classification and Examples of T-norms

It is called *Continuous*, if it is so as a function, when we consider the usual interval topology on [0, 1] x [0, 1].

In a similar way, we may define left-continuous and right-continuous.

It is called *Archimedean,* if it has the Archimedean property, i.e. if for each values x and y, that belongs to the open unit interval, (0, 1), there is a natural number, *n,* such that

$$x * x * \ldots^{n)} * x \geq y$$

Some illustrative examples of T-norms will be:

- *Minimun T-norm,* also called the *Gödel T-norm.* It is the pointwise largest T-norm.

- *Product T-norm.* It is the ordinary product of real numbers, being a strict Archimedean T-norm.
- *Lukasiewicz T-norm.* It will be a nilpotent Archimedean T-norm, being pointwise smaller than the aforementioned product T-norm.
- *Drastic T-norm.* It is the pointwise smallest T-norm, being a right-continuous Archimedean T-norm.
- *Nilpotent minimum.* It constitutes an example of left-continuous, but not continuous, T-norm. Note that despite its name, it is not a nilpotent T-norm.
- *Hamacher product.* It will be a strict Archimedean T-norm.

Recall that:

A continuous T-norm is *Archimedean* if and only if 0 and 1 are its only idempotent elements.

And it is *Nilpotent* if and only if each $x < 1$ is a nilpotent element of T.

For each continuous T-norm, the set of their idempotent elements will be a closed subset of [0, 1].

5. Introduction to T-conorms

They are a dual concept of T-norm, and also may be called *S-norm;* abridgedly, *S.*

Because under the order-reversing operation

$$x \rightarrow 1 - x$$

on the closed unit interval, [0, 1], we have the transformation

$$T(x, y) = 1 - S(1 - x, 1 - y)$$

Therefore, every T-norm can be generated from a S-norm, and vice versa.

Indeed, it is a generalization of De Morgan´s Laws.

It verifies the following conditions:

1) Monotonicity: if $a \leq c$ and $b \leq d$, then $S(a, b) \leq S(c, d)$.
2) Commutativity: $S(a, b) = S(b, a)$.
3) Associativity: $S(a, S(b, c)) = S(S(a, b), c)$.
4) Existence of identity element: $S(a, 0) = a$.

S-norms are used to represent union, in fuzzy set theory.

And also to represent logical disjunction, in fuzzy logic.

6. Examples of S-norms, or T-conorms

They are duals to fundamental T-norms. So, for instance [4, 7],

- *Maximum S-norm,* dual to the minimum T-norm. It will be the smallest S-norm.
- *Probabilistic Sum.* It is the dual to the product T-norm. Working on Probability Theory, it expresses the probability of the union of independent events.
- *Bounded Sum.* It is the dual of Lukasiewicz T-norm. Correspondingly, it will be the standard semantics for strong disjunction in Lukasiewicz fuzzy logic.
- *Drastic S-norm.* It is the dual of drastic T-norm. And it will be the largest S-norm.
- *Nilpotent maximum.* It is the dual of nilpotent minimum.
- *Einstein Sum.* It is the dual of Hamacher T-norm.

7. Introduction to Copulas

We consider two-dimensional copulas [4, 7]. It will be denoted by *C.*

We describe this mathematical object as a function

$$C: [0, 1]^2 \rightarrow [0, 1]$$

with these properties:

1) $C(0, x) = C(x, 0) = 0$
2) $C(1, x) = C(x, 1) = x$, for any $x \in [0, 1]$
3) C is *2-increasing,* i.e. for all $x, y, x', y' \in [0, 1]$, with $x \leq x'$, and $y \leq y'$, for the volume (denoted by V_C) of the respective rectangle

$$[x, x'] \times [y, y']$$

it holds

$$V_C([x, x'] \times [y, y']) = C(x, y) + C(x', y') - C(x, y') - C(x', y) \geq 0$$

8. Note

An interesting application of Copulas may be the *Sklar´s Theorem,* according to which [7]:

For each random vector, (X, Y), will be characterized by some copula, C, in a way that for its joint distribution, F_{XY}, and for the corresponding marginal distributions, F_x and F_Y, we have

$$F_{XY}(x, y) = C(F_X(x), F_Y(y))$$

It would be a mere sample of many other possible applications of all these mathematical techniques, also interesting from a mathematical viewpoint. For these reasons, it is currently a theoretical field quickly evolving.

9. **References**

[1] Bergmann, M. (2008). *An Introduction to Many-Valued and Fuzzy Logic: Semantics, Algebra, and Derivation Systems.* Cambridge University Press
[2] Garrido, A. (2004). Logical Foundations of Artificial Intelligence. *Proc. FotFS V (Foundations of the Formal Sciences),* Institut für Angewändte Mathematik, Friedrich-Wilhelm Universität, Bonn
[3] Hangos, K. (2008). *Fuzzy rule-based systems. Fuzzy sets and calculus.* Department of Computer Science. PhD course. University of Veszprém, Hungary, 2008.
[4] Klement, E. P., Mesiar, R. and Pap, E. (2000). *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000.
[5] Klir, G. and Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic, Theory and Applications.* Prentice-Hall
[6] Mendel, J. M. (2001). *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions.* Prentice Hall PTR.
[7] Pap, E., Takács, M.(2006), Two-Dimensional Copulas as Important Binary Aggregation Operators. JAC III 10 (4): 522-526.
[8] Ross, T. (2004). *Fuzzy Logic with Engineering Applications.* Wiley.
[9] Tanaka, K. and Niimura, T. (2007). *An Introduction to Fuzzy Logic for Practical Applications*, Springer.
[10] Zadeh, L. (1965). Fuzzy Sets. *Information and Control,* 8: 338-55.