Approximation Solutions of Boundary-contact Problems of Non-classical Diffusion Models Coupled-elasticity

Manana Chumburidze
Akaki Tsereteli State University
22/6 str. K.Meskhi, Kutaisi, Georgia, 4600
maminachumb02@gmail.com

David Lekveishvili
Akaki Tsereteli State University
59 Str.Tamar Mephe , Kutaisi, Georgia, 4600
david.lekveishvili@atsu.edu.ge

Elza Bitsadze
Phd student of Georgian technical University
22/74 str.Javakhishvili, Kutaisi, Georgia, 4600
eliza-bitsadze@mail.ru

Abstract

In this article we formulate and analyze a class of diffusion PDE models for oscillation systems of coupled–elasticity in 2-D bounded domains. Approximate method in Green-Lindsay formulation with thermal and diffusion relaxation times has been developed. Basic Boundary-contact problems for isotropic inhomogeneous finite and infinite media with the inclusion of piecewise elastic material in assumptions that surface is sufficiently smooth have been investigated. The tools applied in this development are based on singular integral equations, Laplace transform, the potential method, Green's Tensors and generalized Fourier series analysis.

Keywords: Approximate method, Boundary-contact problems, coupled thermo-diffusion

1. Introduction

Many methods in the theory of non-classical thermo-elasticity require to the solution of boundary-contact problems (BCPs). A great attention is payed to the construction of solutions in the form that admit efficient numerical evaluation (Chumburidze, 2014; Kupradze, 1983). In this work, a numerical approximation for the solution of BCPs for 2-D oscillation systems coupled thermo-elastic diffusion materials (Chumburidze, 2017) with thermal and diffusion relaxation times has been developed. In particular, the problem is investigated for isotropic piecewise inhomogeneous elastic materials with sufficiently smooth surfaces. Solutions for the finite domain when oscillations are not equal to the natural frequencies and for infinite domain in assumptions that solutions satisfy of radiation conditions in infinite have been constructed.

Algorithms of numerical solution have been obtained for particular cases of boundary-contact conditions when the couple-stresses components, displacement components, rotation, heat flux and temperature, concentration and chemical potential are represented on the surface of Holder class.

Throughout of paper we introduce the following notations: E^2 two-dimensional Euclidean space,

$$x=(x_j);\ y=(y_j);\ j=1,2 \text{ - points of this space,}$$

$$D^{(0)} \text{ is infinite domain with inclusion another elastic material } D^{(1)}$$

$D^{(1)}$ $(D^{(1)} \subset E^2, D^{(0)} = E^2 \backslash \overline{D^{(1)}})$ bounded by the close surface $S \in L_{(2)}(\alpha)$, $\alpha > 0$ with outward positive normal vector.

Investigation of pseudo oscillation systems of generalized coupled thermo-diffusion model for 2-D isotropic homogeneous elastic materials in the Green-Lindsay (Green, Lindsay, 1972) formulation is presented in (Chumburidze, 2014).

Let us consider isotropic inhomogeneous elastic materials. In this case, in order of results in publications (Chumburidze, 2016; Burchuladze, 1985) the following mathematical model has been obtained:

$$(\mu_r+\alpha_r)\Delta u(x,\tau)+(\lambda_r+\mu_r-\alpha_r)graddivu+2\alpha_r rotu_3-\sum_{i=1}^{2}\gamma_{i\tau r}\,gradu_{i+3}-\varrho_r\tau^2u=\Gamma^{(1r)}(x)$$

$$(\nu_r+\beta_r)\Delta u_3(x,\tau)+2\alpha_r rotu-4\alpha_r u_3-I_r\tau^2u_3=\Gamma_{3r}(x) \qquad (1)$$

$$\aleph_{1r}\Delta u_4(x,\tau)-a_{1r}\tau u_4-a_{12r}\tau u_5-\gamma_{1r}\tau divu=\Gamma_{4r}(x)$$

$$\aleph_{2r}\Delta u_5(x,\tau)-a_{2r}\tau u_5-a_{12r}\tau u_4-\gamma_{2r}\tau divu=\Gamma_{5r}(x)$$

where $u=(u_1,u_2)$ is a displacement vector, u_3 is a characteristic of rotation, u_4 is a temperature variation, u_5 is a chemical potential,

$$rotu_3=\left(\frac{\partial u_3}{\partial x_2},-\frac{\partial u_3}{\partial x_1}\right)',rotu=\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right),$$

$\gamma_{\tau rk}=\gamma_{rk}(1+\tau_{1r}\tau),\ a_{\tau rk}=a_{rk}(1+\tau_{0r}\tau),\tau_{1r}>\tau_{0r}>0$ –constants of relaxation, $\varrho_r>0,\mu_r>0,\alpha_r>0,\ 3\lambda_r+2\mu_r>0,\nu_r>0,\beta_r>0,I_r>0,\gamma_{\tau rk}>0,a_{\tau rk}>0,k=1,2,\ a_{1r}a_{2r}-a_{12r}^2>0$

- constants of elasticity of $D^{(r)}$ domains (r=0,1), Δ is a two-dimensional Laplacian operator; $\tau=\sigma+iq,\ \sigma>0$ (corresponds to the general dynamical problems) (Chumburidze, 2016; Sherief, 2004),

$\Gamma^{(r)}=(\Gamma^{(1r)},\Gamma_{3r},\Gamma_{4r},\Gamma_{5r})=(\Gamma_{1r},\Gamma_{2r},\Gamma_{3r},\Gamma_{4r},\Gamma_{5r})\subset C^{0,\alpha}(D^{(r)}),\ \alpha>0$ are the given vectors.
are the given vectors.

Let us construct matrices of generalized stress operators of coupled thermo-elasticity in $D^{(r)}$ domains:

$$R^{(r)}(\partial x,n(x))=\left|\begin{matrix}\left|T^{(r)}(\partial x,n(x))\right|_{3x3} & \left|-N(x)\sum_{i=1}^{2}\gamma_{i\tau r}\right|_{3x2}\\ \multicolumn{2}{c}{\left|\delta_{5q}\frac{\partial}{\partial n}\right|_{1x5}}\end{matrix}\right|_{5x5} \qquad (2)$$

Where $q=\overline{0,3};N(x)=(n,0,0),n=(n_1,n_2)$,

$T^{(r)}(\partial x,n(x))=\left|T_{jk}^{(r)}(\partial x,n(x))\right|_{3x3}$-matrices of stress operators on the plain *(Chumburidze,2014):*

$$T_{jk}^{(r)}(\partial x,n(x))=\lambda_r n_j(x)\frac{\partial}{\partial x_k}+(\mu_r-\alpha_r)n_k(x)\frac{\partial}{\partial x_j}-(\mu_r+\alpha_r)\delta_{kj}\frac{\partial}{\partial n(x)},j,k=1,2\ ;$$

$$T_{jk}^{(r)}(\partial x,n(x))=-2\alpha_r\sum_{p=1}^{2}\xi_{jkp}\,n_p(x),j=1,2,k=3;$$

$$T_{jk}^{(r)}(\partial x,n(x))=(\nu_r+\beta_r)\delta_{kj}\frac{\partial}{\partial n(x)},j=3,k=1,3.$$

The matrices of (1) pseudo oscillation systems have the form:

$$L^{(r)}(\partial x,\tau)=\begin{vmatrix} \left|L^{(1r)}-\varrho_r\tau^2\right|_{2x2} & \left|L^{(2r)}\right|_{2x1} & \left|-\gamma_{1r\tau}G^T(\partial x)\right|_{2x1} & \left|-\gamma_{2r\tau}G^T(\partial x)\right|_{2x1} \\ \left|L^{(3r)}\right|_{1x2} & L^{(4r)}-I_r\tau^2 & 0 & 0 \\ \left|-\gamma_{1r}\tau G(\partial x)\right|_{1x2} & 0 & \Delta-\dfrac{\tau a_{1r}}{\aleph_{1r}} & -\dfrac{\tau a_{12r}}{\aleph_{1r}} \\ \left|-\gamma_{2r}\tau G(\partial x)\right|_{1x2} & 0 & -\dfrac{\tau a_{12r}}{\aleph_{2r}} & \Delta-\dfrac{\tau a_{2r}}{\aleph_{2r}} \end{vmatrix}_{5x5}$$

Where

$$L_{ij}^{(1r)}(\partial x)=\delta_{ij}(\mu_r+\alpha_r)\Delta+(\lambda_r+\mu_r-\alpha_r)\frac{\partial^2}{\partial x_i\partial x_j},L_{ij}^{(2r)}(\partial x)=L_{ij}^{(3r)}(\partial x)=-2\alpha_r\sum_{p=1}^{2}\xi_{ijp}\frac{\partial}{\partial x_p},$$
$$L_i^{(4r)}(\partial x)=\left((\nu_r+\beta_r)\Delta-4\alpha_r\right),G(\partial x)=(\partial x_1,\partial x_2)$$

Where $\delta_{ij}$is Kronecker's symbol, ξ_{ijp} is Levi-Chivita's symbol.

Therefore, (1) can be written in the form:

$$L^{(r)}(\partial x,\tau)U(x,\tau)=\Gamma^{(r)}(x)\quad(3)$$

2. Approximate Solutions of BBCP

In our investigations, we consider BBCP of pseudo oscillation problems in assumption that τ^2 are not equal to the natural frequencies in internal problems and satisfy of radiation conditions for external problems (Chumburidze,2016, Eshkuvatov, 2009):

$$u_j(x)=O\left(\frac{1}{|x|}\right),\qquad \frac{\partial u_j(x)}{\partial x_k}=o\left(\frac{1}{|x|}\right),\qquad k=1,2;j=\overline{1,4},$$

3. Statement problem. It is required to find regular solutions $U(x) = (u, u, u_4, u_5)$ with thefollowing conditions:

$$\forall\,x\in D^{(r)},r=0,1:\qquad L^{(r)}(\partial x,\tau)U(x)=\Gamma^{(r)}(x)$$
$$\forall z\in S\in L_2(\alpha),\alpha>0:\ \ \{U(z)\}^- -\{U(z)\}^+=f(z)\qquad(4)$$
$$\{R^{(0)}(\partial z,n)U(z)\}^- -\{R^{(1)}(\partial z,n)U(z)\}^+=g(z)$$

Where $L^{(r)}(\partial x)-$ matrices of static systems ,$f\in C^{1,\beta}(S)$, $g\in C^{0,\beta}(S)$, $S\in L_{(2)}(\alpha)$, $0<\beta<\alpha\leq 1$, symbols "+" and "–" signs the boundary values of functions in domains $D^{(1)}$ and $D^{(0)}$ correspondingly.

The existence and uniqueness of this solution has been proved in (Chumburidze, 2014), (Kupradze, 1983).

4. Solution

Solution of the **Problem** will be found by use the formula of regular solutions (Chumburidze, 2014), (Constanda, 2014). Take in account of boundary-contact conditions (4) we will get:

$$2U(x)=-\int_S\Phi^{(0)}(x-y,\tau)\{R^{(0)}U\}^-d_yS+\int_S\left[R^{(0)}\Phi^{(0)}(x-y,\tau)\right]'\{U\}^-d_yS-F_{(0)}(x),\ \ x\epsilon D^{(0)}\ (5)$$

$$0=\int_S\Phi^{(0)}(x-y,\tau)\{R^{(0)}U\}^-d_yS-\int_S\left[R^{(0)}\Phi^{(0)}(x-y,\tau)\right]'\{U\}^-d_yS-F_{(0)}(x),\ \ x\epsilon D^{(1)}\quad(6)$$

$$2U(x)=\int_S\Phi^{(1)}(x-y,\tau)\{R^{(0)}U\}^-d_yS-\int_S\left[R^{(1)}\Phi^{(1)}(x-y,\tau)\right]'\{U\}^-d_yS+F_{(1)}(x),\ x\epsilon D^{(1)}\quad(7)$$

$$0=\int_S \Phi^{(1)}(x-y,\tau)\,\{R^{(0)}U\}^{-}d_yS-\int_S[R^{(1)}\Phi^{(1)}(x-y,\tau)]'\{U\}^{-}d_yS+F_{(1)}(x),\quad x\epsilon D^{(0)} \tag{8}$$

Where

$$F_{(1)}(x)=-\int_{D^1}R^{(1)}(x-y)\,\Gamma^{(1)}(y)d_yS+\int_S[R^{(1)}\Phi^{(1)}(x-y,\tau)]'f(y)d_yS-\int_S\Phi^{(1)}(x-y,\tau)\,g(y)d_yS$$

$$F_{(0)}(x)=\int_{D^0}\Phi^{(0)}(x-y,\tau)\,\Gamma^{(0)}(y)d_yS$$

$\Phi^{(r)}(x-y,\tau)$- fundamental solution of (3) system (*Chumburidze,2014, JIANG, 2011)*. Symbol $'$ sign transpose of a matrix.

Allow us consider the matrices:

$$H_{(r)}(x,y)=\left\|[R^{(r)}\Phi^{(r)}(x-y,\tau)]',-\Phi^{(r)}(x-y,\tau)\right\|_{5x10},\forall x\in D^{(r)},y\epsilon S$$

And a vector of ten components:

$$\lambda(y)=\left(\{U\}^{-},\{R^{(0)}U\}^{-}\right)'$$

Then (6) and (8) to get the following form:

$$\int_S H_{(1)}(x,y)\lambda(y)d_yS=F_{(1)}(x),\qquad \forall x\epsilon D^{(0)} \tag{9}$$

$$\int_S H_{(0)}(x,y)\lambda(y)d_yS=F_{(0)}(x),\qquad \forall x\epsilon D^{(1)} \tag{10}$$

Let us construct auxiliary domains $\overline{D^{r}}$ bounded by the closed surfaces $\overline{S^{r}}$ of Holder class (*Chumburidze,2016)* in the following assumptions:

$\overline{D^{(1)}}\subset\overline{D^{(1)}}\subset\overline{D^{(0)}}\subset D^{(0)}$ and $\left\{x_j^{(r)}\right\}_{j=1}^{\infty}\in\overline{S^{(r)}}$ $(r=0,1)$ are everywhere accounted set of points.

Let us insert points: $x_j^{(0)}\in\overline{S^{(0)}}$ and $x_j^{(1)}\in\overline{S^{(1)}}$ in (9) and (10) correspondingly, then we will get:

$$\int_S H_{(1)}\left(x_j^{(0)},y\right)\lambda(y)d_yS=F_{(1)}(x_j^{(0)}) \tag{11}$$

$$\int_S H_{(0)}\left(x_j^{(1)},y\right)\lambda(y)d_yS=F_{(0)}(x_j^{(1)}),\ (j=1,2,\dots) \tag{12}$$

In left side of equations (11) and (12) we have scalar multiplications of $\lambda(y)$ on the accounted set of vectors(*Chumburidze,2017)*:

$$\left(R^{(1)}\Phi_p^{(1)}\left(y-x_j^{(0)},\tau\right),-\Phi_p^{(1)}\left(y-x_j^{(0)},\tau\right)\right),\left(R^{(0)}\Phi_p^{(0)}\left(y-x_j^{(1)},\tau\right),-\Phi_p^{(0)}\left(y-x_j^{(1)},\tau\right)\right), \tag{13}$$

$$\left(p=\overline{1,5};j=1,2,..\right),y\in S$$

And on the right side of same equations (11) and (12) we have vectors which are known.

Let us consider accounted set of vectors:

$$\{H^{(k)}(x_j,y)\}_{j=1}^{\infty},k=\overline{1,10} \tag{14}$$

Where

$$H^{(k)}(x_j,y)=\begin{cases}H_{(1)}\left(x_j^{(0)},y\right),if\ \ k=\overline{1,5};\\ H_{(0)}\left(x_j^{(1)},y\right),if\ \ k=\overline{6,10};\end{cases} \tag{15}$$

Allow us make numbering of elements in (14) by the following form:

$$\varphi^{(k)}=\ H^{(l_k)}(x_{[(k+9)/10]},y),\,l_k=k-10\left[\frac{k-1}{10}\right] \tag{16}$$

The next theorem is proved there:

Theorem. Accounted set of vectors $\{\varphi^{(n)}(y)\}_{n=1}^{\infty}$,is linearly independent and full in the space $L_2(s)$

Proof:

Let us define the constants $a_n(n=1,..,N)$ from the conditions of minimization Mean-squared norm:

$$\left\|\lambda(y)-\sum_{n=1}^{N} a_n\varphi^{(n)}(y)\right\|_{L_2(s)}=\min \quad (17)$$

So, a_n are sought by the solution the system:

$$\sum_{n=1}^{N} a_n\int_s(\varphi^{(n)},\varphi^{(k)})ds=\int_s(\lambda,\varphi^{(k)})ds, k=1,..N.$$

where N is any number. In order (16) we have:

$$\sum_{n=1}^{N} a_n\int_s(\varphi^{(n)},\varphi^{(k)})ds=\int_s\left(\lambda,\ H^{(l_k)}\left(x_{[(k+9)/10]},y\right)\right)ds, k=\overline{1,N}$$

However accordingly (11), (12) and (15) we have:

$$\int_S H^{(k)}(x_j,y)\,\lambda(y)d_yS=F_k(x_j),\qquad k=\overline{1,10}.$$

Hence $a_n$should be sought from the following system:

$$\sum_{n=1}^{N} a_n\int_s(\varphi^{(n)},\varphi^{(k)})ds=F_{l_k}\left(x_{[(k+9)/10]}\right), k=1,..N \quad (18)$$

Take in account the property of linearly dependent vectors we can discuss that (18) system is uniquely solvable(*Chumburidze,2016)* .

Allow us construct the following vectors:

$$U_N=\frac{(-1)^{r+1}}{2}\left\{\int_S H_{(r)}(x,y)\sum_{k=1}^{N} a_k H^{(l_k)}\left(x_{[(k+9)/10]},y\right)d_ys+F_{(r)}\right\}\qquad (19)$$

We should prove that U_N are approximate solutions.

Indeed in order (5) and (7) we have:

$$2U=(-1)^{r+1}\left\{\int_S H_{(r)}(x,y)\lambda(y)d_ys+F_{(r)}\right\}$$

By using Cauchy-Bunyakovski inequality (*Chumburidze,2017)* we have:

$$|U-U_N|=\left|\int_S H_{(r)}(x,y)\left(\lambda(y)-\sum_{k=1}^{N} a_k H^{(l_k)}\left(x_{[(k+9)/10]},y\right)\right)d_ys\right|\le$$

$$\le\left\{\int_S|H_{(r)}(x,y)|^2d_ys\right\}^{1/2}\left\{\int_S\left|\lambda(y)-\sum_{k=1}^{N} a_k H^{(l_k)}\left(x_{[(k+9)/10]},y\right)\right|^2d_ys\right\}^{1/2}$$

Where

$\int_S|H_{(r)}(x,y)|^2d_ys$ functions are bounded (*Chumburidze, 2014 Orlando, 1985).* Also according to (17) we have:

$$\lim_{N\to\infty}\int_S\left|\lambda(y)-\sum_{k=1}^{N} a_k H^{(l_k)}\left(x_{[(k+9)/10]},y\right)\right|^2d_ys=0$$

Therefore:

$$\lim_{N\to\infty}|U-U_N|=0$$

5. Conclusion

Thus, approximate solutions of BBCP by using the boundary integral method and generalized Furrier series analysis for an infinite domain with inclusion another elastic material have been constructed.

As a result, it will be shown that same method is reliable for obtaining numerical approximation for infinite domain with inclusion of several elastic materials.

References

Chumburidze, M. (2014). Non-classical models of the some theory of boundary value problems. Saarbrcken, Germany.

Chumburidze, M., & Lekveishvili, D. (2017). Numerical approximation of basic boundary-contact problems, DOI 10.1115/DETC2017-67097, ASME, PP. V009T07A021; 7 pages.

Chumburidze, M., & Lekveishvili, D. (2014). Approximate Solution of Some Mixed Boundary Value Problems of the Generalized Theory of Couple-Stress Thermo-Elasticity. World Academy of Science, engineering and Technology, 6(28), 161-163.

Constanda, C. (2014). Generalized Fourier Series. Mathematical Methods for Elastic Plates, Springer.

Chumburidze, M. (2016). Approximate Solution of Some Boundary Value Problems of Coupled Thermo-Elasticity.Mathematical and Computational Approaches in Advancing Modern Science and Engineering, Springer, DOI 10.1007/978-3-319-30379-6_7, pp 71-80.

Kupradze, V. D., Gegelia, T. G., Basheleishvili, M. O., & Burchuladze, T. V. (1983). Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermo Elasticity. North-Holland, Amsterdam-New York.

Eshkuvatov, Z., & Long, N. (2009). Approximate solution of singular integral equations of the first kind with Cauchy kernel, Applied Mathematics Letters doi:10.1016/j.aml.2008.08.001.

Mathon, R., Johnston, R. L. (1977). The approximate solution of elliptic boundary-value problems by fundamental solutions, SIAM Journal on Numerical Analysis. 638–650.

Chen, W., Wang, F. Z. (2010). A method of fundamental solutions without fictitious boundary, Engineering Analysis with Boundary Elements. 34 530–532.

Jiang, X. & Chen, W. (2011). Method of fundamental solution and boundary knot method for helmholtz equations: a comparative study (Chinese). Chinese Journal of Computational Mechanics. 28, 3338–344.

Ezzat, M., Zakaria, M. (2004). Generalized thermoelasticity with temperature dependent modulus of elasticity under three theories. J. Appl. Math. Comput. 14, 193–212.

Green, E., Lindsay, K.A. (1972). Thermoelasticity. J. Elast. 2, 1–7.

Orlando, F. (1985). Hankel Functions. Mathematical Methods for Physicists, 3rd edn. Academic,Orlando.

Rezazadeh, G., Vahdat, A. (2012). Thermoelastic damping in a micro-beam resonator using modified couple stress theory. Acta Mechanica, Springer 223(6), 1137–1152.

Sherief, H., Hamza, F., Saleh, H. (2004). The theory of generalized thermoelastic diffusion. Int. J. Eng 5, 591–608.

Hazewinkel, M., (ed.) (2001). Fundamental solution, Springer.

Ezzat, M., Zakaria, M. (2004). Generalized thermoelasticity with temperature dependent modulus of elasticity under three theories. J. Appl. Math. Comput. 14, 193–212.

Burchuladze, T., Gegelia, T. (1985). Development of the Potential Method in Elasticity Theory.Mecniereba, Tbilisi.

Dr. **Manana Chumburidze** has obtained PhD degree in Mathematics sciences in Ivane Javakhishvili Tbilisi State University of Georgia in 1999. She is Associate Professor of Department Computer Sciences at Akaki Tsereteli State University of Georgia. She is membership of The American Society of Mechanical Engineers (ASME) and member of the Scientific and Technical Committee of Editorial Review Board on Mechanics and Computational Sciences in the WASET (World Academy of Science, Engineering and Technology) organization of USA.

Dr. **David Lekveishvili** has obtained PhD degree in Mathematics sciences in Lomonosov Moscow State University in 1983. He is Dean of Faculty Exact and Natural Sciences and Associate Professor Department of Mathematics at Akaki Tsereteli State University of Georgia. Research interests include Applied Mathematics, partial differential equations, computer science. He is author several articles and active participant different international conferences in the World Academy of Science, Engineering and Technology.

Elza Bitsadze is a Phd student of Georgian technical University and teacher of computing of Department Computer Science at Akaki Tsereteli State University of Georgia. Her research interest is computer sciences. She is author several articles and active participant different international conferences.