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# Mathematical Creativity: A Peircean Abductive Proposal

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**Abstract:** *This paper explores mathematical creativity through the lens of Charles Sanders Peirce's concept of abductive reasoning. Peirce, who defined mathematics as the study of pure hypotheses and their consequences, saw abduction as the process of forming explanatory hypotheses, the only logical operation that introduces new ideas. We argue that mathematical creativity, with its focus on generating novel and meaningful ideas, patterns, and solutions, is intrinsically linked to abduction and takes place through diagrammatic reasoning. Mathematicians, faced with surprising facts or observations, engage in abductive reasoning to propose new theorems or conjectures that explain these phenomena. This creative leap, going beyond deductive and inductive reasoning, lies at the heart of mathematical discovery. We illustrate this process with examples from contemporary mathematics, including Wiles' proof of Fermat's Last Theorem and Perelman's resolution of the Poincaré Conjecture. Furthermore, we connect Peirce's notion of diagrammatic reasoning to mathematical creativity, highlighting the role of experimentation and observation in this process. Finally, we link our approach to Fernando Zalamea's advocacy for a "synthetic philosophy of contemporary mathematics" that engages directly with the creative practices of mathematicians.*

**Keywords:** *mathematical reasoning; diagrammatic reasoning; abduction; Charles Sanders Peirce; contemporary mathematics.*

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> “[It] is hardly credible however that there is anybody who does not know that mathematics calls for the profoundest invention, the most athletic imagination, and for a power of generalization in comparison to whose everyday performances the most vaunted performances of metaphysical, biological, and cosmological philosophers in this line seem simply puny” (Peirce, 1893–1903/1933, CP 4: 611).

**1. Peirce’s mathematics, creativity and abduction**

Charles S. Peirce defines mathematics as “the study of pure hypotheses and their consequences” (Peirce, 1931–1958, MS 303, 1903; Peirce, 1998, pp. 144, 146). This means that mathematics is the science of possibilities (not actualities) and, therefore, it is a science not constrained by experience of facts, namely, it is not a positive science. This may contradict the popular vision of mathematics as a science of necessary truths and deductive methods of inference; this even may contradict his father's definition, the mathematician of Harvard, Benjamin Peirce, according to which mathematics is “the science that draws necessary conclusions”. Although certainly those axioms or necessary truths and deductive apparatuses are the results of certain mathematical creations, the essence of mathematics is the creative process by which those processes were discovered. For instance, euclidean geometry has been considered popularly as a necessary truth without contradiction; but it depends on five postulates that Euclides freely contemplated; if mathematicians start to freely play with the fifth postulate, for instance, and instead accept that only one straight line can touch a point in another straight line, they accept that no point can be touched, or several can be touched, then several other non-euclidean geometries would be envisioned as possibilities. That Euclides’ fifth postulate is just one hypothesis chosen by him is evident by the fact that it presupposes a plane bidimensional surface; but such a postulate must change if some other surfaces, such as a sphere or any other curve, are presupposed.

Peirce also locates mathematics, in the system of all sciences, as the very first science, as the ground for philosophy and positive sciences. This implies that mathematics is a free science, not subordinate to any other, and it is open to all possibilities. It is not constrained by experience nor by the principles of any other science. So mathematics is the freest of all. Instead, mathematics provides the first principles for any other science. Mathematical hypotheses have an important impact on the development of other sciences. That is why the mathematical continuum (the idea that there is unity across all branches of mathematics) can change drastically the way we do philosophy and other sciences; Synechism (the metaphysical doctrine that gives priority to continuity over discreteness) as a philosophical doctrine can overcome the nominalistic and dualistic presuppositions that ground most of the philosophical tradition. Zalamea, for instance, foresees that many other hypotheses of contemporary and highly technical mathematics can transform the way philosophy is still made (2012).

Mathematical creativity requires more than tracking consequences or generalising particular facts. At its core, mathematical creativity involves generating novel and meaningful ideas, patterns, or solutions as well as visualisations, diagrams, and relationships. This process often entails going beyond the known, venturing into uncharted possibilities of thought, and using our imaginative powers to create within a world of possibilities, as the world of mathematical relations is. However, how this takes place demands an understanding of how we think, and more specifically, how is it possible to arrive from knowledge of what we know to information that we did not know: we call this activity “inference.” Having reflective control of our mathematical inferences gives us reflective clarity as to how to improve our practices and understand the creative process involved.

Now, inference takes place in different ways, we can depart from information that we know generally to information that applies, through a rule to particular cases. In this latter sense, we are operating deductively. Deductive inferences are based on first premises and first principles that must be given beforehand; so deductive inferences and the sciences that use them are subordinated

to other sciences that provided them those principles. Deductive sciences must be grounded in sciences more fundamental and therefore, they are not free but dependent on those other previous sciences. Deductive inferences are the less creative kind of inferences, for they only make explicit the information that was already claimed in the premises. On the contrary, mathematical inference must be free of any constraints and must be on the grounds of any other science. As a science of pure hypothesis, mathematics must be grounded in an inferential process different from deduction. The rigid process of deduction would not allow mathematics to envision new hypotheses; it would simply track the necessary consequences of a given postulate, axiom, or any other hypothesis.
We can also depart from cases and generalise what we learn through the statistical repetition or frequency of instances of information (cases) that eventually will support a general tendency, this kind of inference is called induction. However, mathematics can not either be grounded in induction because the innovative process of creating a new hypothesis does not proceed simply for the generalisation of a sample. The new and innovative hypotheses are different from the experienced sample. One can see falling apples millions of times without imagining a general force of attraction. An induction would say simply that “all apples fall down”, but a new hypothesis explains why they fall.

Consequently, when we want to understand mathematical creativity these two modes of inference fall short in many ways: we need a mode of inference that can help us to have ampliative understanding and creative ideas in the world of possibilities: deduction brings information already assumed; induction reports probability and statistical information; neither of these help to bring “new” information that gives the inquirer “reflective clarity” and a “unifying hypothesis” that helps us retrieve reflective control in our mathematical inquiries. The mode of inference that pertains to such activities is what Charles Sanders Peirce called “abductive” reasoning. Peirce's concept of abductive reasoning, also known as '*hypothesis*' or '*retroduction*,' serves as a powerful lens through which to understand this creative process.

Abduction, in Peirce's words, is "the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new idea" (Peirce, 1931–1958, CP 5.171). It begins with a surprising fact or observation, a phenomenon that does not fit neatly into existing frameworks. The mathematician then proposes a hypothesis, a new idea or rule, that would, if true, explain this surprising fact, it makes it “a matter of course”: it is a reconnection with explanatory continuity. Abduction, therefore, is the mode of inference that postulates a hypothesis that connects a statement of generality with a particular instance through the proposition of a circumstance or case that links the two.

This mode of inference describes the process of mathematical discovery. A mathematician might observe a pattern in numbers or geometric shapes, something unexpected or intriguing. To explain this observation, they might abduce a new theorem or conjecture, a creative leap that potentially opens up new avenues of mathematical exploration. As Peirce notes, abduction is not merely explanatory, it is also about generating new ideas. He highlights this when he writes about Kepler's work:

> For example, at a certain stage of Kepler’s eternal exemplar of scientific reasoning, he found that the observed longitudes of Mars, which he had long tried in vain to get fitted with an orbit, were (within the possible limits of error of the observations) such as they would be if Mars moved in an ellipse. The facts were thus, in so far, a likeness of those of motion in an elliptic orbit. Kepler did not conclude from this that the orbit really was an ellipse; but it did incline him to that idea so much as to decide him to undertake to ascertain whether virtual predictions about the latitudes and parallaxes based on this hypothesis would be verified or not. This probational adoption of the hypothesis was an Abduction. (Peirce, 1931–1958, CP 2.96, c. 1902).

Kepler’s operation was an abduction. It was the inference of a minor premise, not the inference of a conclusion. Kepler did not conclude from the theory that Mars would move in an ellipse; but from the fact that Mars moved in an ellipse, he concluded that the theory was true. In this sense, the mathematician does not deduce a new result from existing knowledge but rather creatively proposes a new principle that accounts for the observed phenomena. This abductive leap is at the heart of mathematical creativity, driving the field forward into novel and exciting directions not part of the previous information.

What we have argued does not mean that mathematics as a science is entirely hypothetical. Abduction is at the heart of mathematics. Its lack of empirical constraints makes mathematics benefit more from hypotheses than any other science, perhaps at the same level as the arts. Creativity is as important in mathematics as it is in art (Anderson, 1987). Nonetheless, deduction and induction play, as in any other science, an important and methodological role. After all, mathematics is also the study of the consequences of the pure hypothesis. Once a hypothesis has been postulated, scientists have to track the necessary consequences that derive from it. When Euclides accepts the five postulates, he can derive an enormous quantity of theorems; and when someone hypothesises that other postulates are possible, he or she can derive a bunch of many other theorems that will constitute a non-euclidean geometry. The tracking of necessary consequences of any hypothesis is deduction. This second step has been considered popularly the main character of mathematics, but it is only a secondary one that depends on the visionary and revolutionary new hypothesis.

After the consequences are tracked deductively, the next step in Peirce's methodeutics is to corroborate those consequences in experiences inductively. Since mathematics is not a positive science, that is perhaps the reason Peirce did not include induction into the definition of mathematics. Although mathematics does not need to corroborate its results in experience, it can still benefit from generalising and accumulating evidence in favour. For instance, one of the consequences that can be derived from Euclid's five postulates is that the internal angles of a triangle are equivalent to two right angles. It is a deductive consequence based entirely on formal inferences; it does not require measuring each possible triangle in the observed universe. Nonetheless, if a teacher can prove it to his or her students by adding the angles of, let's say, twenty different triangles on the blackboard, each result will corroborate the same results and the theorem will be more convincing to any skeptical student.

Mathematical creativity is, therefore, intrinsically linked to abductive reasoning. It is predicated on the ability to observe the unexpected, and then creatively propose new ideas and hypotheses to explain these observations. As Peirce aptly puts it, this process is "the only logical operation which introduces any new idea." It is this generation of new ideas, this ability to creatively conjecture and hypothesise, that fuels the dynamic and ever-evolving field of mathematics beyond the recombination of existing knowledge. However, this abductive process, in the particular case of mathematics, happens with the experimentation with ideas in a diagrammatic way. Because of this, we need to explore what Peirce means by Diagrammatic Reasoning. Let us delve into the meaning of "diagrammatic reasoning" in Peirce's philosophy of mathematics.

## 2. Diagrammatic Reasoning and creativity

The process by which we carry out mathematical hypothesis is the experimentation on the possible ideas that give explanatory continuity and hence help us solve a mathematical problem, but in mathematics, this process is ordered logically through a process of experimentation with diagrams of different sorts. Therefore, the process of diagrammatic reasoning is the enacting of abductive inference. Peirce's answer to the problem of how we make sense of mathematical inquiry is closely linked to his belief in true *continua,* i.e*.,* the belief that continuous structures are real and these are prominent in mathematical thought. Peirce denied that a continuum in mathematics is an aggregate or a collection of individuals, which is the point of view of the nominalistic approach. This implies that for Peirce, mathematics is grounded in continuity, rather than in discrete concepts (discrete concepts are atomic, self-contained unities). He proposed conceiving a continuum as a kind of series in which the members are specified after recognising the underlying continuity presupposed by the series; this can be illustrated by the natural number series, whose properties are derived from its continuity, rather than the sum of the properties of individual numbers. The whole's continuity precedes its parts. Thus, for example, in the same natural number series, the properties of the natural numbers come out from the series and not from the particular numbers attached in an orderly way. Peirce thinks that this kind of misleading path leads to the kind of paradoxes of Achilles and the Tortoise, *i.e*., if series are formed of discrete elements, how can we decide which ones of the infinite possible points in between should we consider the elementary ones? Peirce thinks that what is called today "mathematical induction" (in our case, "mathematical inference"), which he called 'Fermatian inference' (Peirce, 1976, *NEM* 3:49), is valid for any collection whose members in one-to-one correspondence with the natural numbers.

For Peirce, the example of Fermatian inference shows that the nominalist is wrong to think that the only valid principles of reasoning about infinity are those that apply to finite collections (Forster, 2011). Now, the way to intellectually come across this kind of fundamental aspect of the continua is by means of diagrammatic reasoning. Reasoning can be defined in terms of inferences, but carrying out inferences means to realise operations that often involve more than one single medium.

For Peirce, Logic is the science that deals with the principles of right reasoning (Peirce, 1896, W 3:244), principles of formulating hypotheses, deducing testable consequences from them, and evaluating their truth or falsehood. This sense of logic is different from the contemporary use; it is an account for the kind of inferences that we can carry out by deduction, induction, and abduction, as we explained above. Diagrammatic representations allow us to carry that kind of reasoning in mathematical expressions, not only deducing but also formulating hypotheses and abducing mathematically by proposing a restoration of the rational path of inference through the means of a hypothesis. As Sara Barrena says: "To our knowledge goes forth [brings out new ideas] it is necessary not only to have good ideas -abductive hypothesis- but also to work with them" (2021; our translation). A diagram is a structure or visual object with which we can work, or what we can act or experiment. An abductive diagram represents a hypothetical state of things that we proceed to directly observe. We can have diagrams of real existing things, as a map or as the chart of a financial report; they can be useful for seeing complex information. However, a diagram is more useful when it represents a hypothetical state of things envisioned by abduction. This means that the object represented in a diagram can be both a real object and an imaginary object. In both cases, "Diagrammatic reasoning implies, therefore, the creation of the diagram —constructing a representation—, observing it, experimenting on it, and right analysis of the results" (Barrena, 2021, our translation. See also NEM 4:47ff., 1902).

A diagram, then, is an object that in abductive reasoning represents the hypothetical state of things the mathematician envisioned; in fact, the creation of the diagram requires the imaginative powers of abduction. Abductive reasoning is not only entertaining a new idea in mind but imagining, i.e. visualising how that new idea would be if it were true. Once the diagram is created, once we have that hypothetical state of things constructed in a diagram, we can start to manipulate it. Once the diagram imagines the new state of things, the necessary continuity is restored to have rational self-control over the understanding of the mathematical issue.

How this process of reasoning and abduction takes place requires us to frame diagrammatic reasoning in the context of inferential abductive knowledge. According to Shin & Lemon (2011), we can consider two kinds of diagrammatic reasoning in which we can observe how we postulate hypothesis within a diagram:

(i) External diagrammatic representations: These are constructed by the agent in a medium in the external world, but are meant as representations by the agent. These include but are not limited to experiments carried upon mathematical structures, for example, topological structures imagined in processes of transformation.
(ii) Internal diagrams or images: these comprise the (controversial) internal representations that are posited to have some pictorial properties. (Chandrasekaran, 1995). These diagrams postulate a particular "missing link" in between representations, for example, when we imagine a

topological structure such as a sphere becoming two spheres, we have to imagine them as coming apart to a critical point of separation in which one object looks like two spheres touching or one sphere becoming two.

A Peirce scholar, Hoffmann (2003), has summarised the process of diagrammatic reasoning found in Peirce's account in the following set of steps (Peirce, 1931–1958), and we included a clause in the second step to clarify a particular aspect of the intentionality within the inquirer and their epistemic capabilities:

1. Constructing a diagram using a given representational system (euclidean geometry, Peirce and Peano Arithmetic, a language, some computer software... etc.) The Construction is motivated by the need to represent relations.
2. Experiment with those diagrams, as long as they define constraints that determine the outcome of experiments and then impinging something that in the actual world will be an inevitable experience.
(a) Creativity in experimenting means modifying representational systems by adding new means to them, deleting some old ones, or reconstructing their systematic order.
3. Observing the results of experimenting, gathering a new insight on getting something out of the outcome of the diagram (unlike a computer, which performs probably better experiments than us, the observations appeal to the idea of a self-controlled conscious inquiry).

The reader can also see that these steps agree with Peirce's method of inquiry. Mathematics cannot track the consequences of a hypothesis in experience for it has no relations with it. But once a diagram is constructed; there is a thing in which he or she can experiment. The need for representing relations is the same need for an explanation that a surprising fact produces; the novel diagram restores the broken continuity of meaning. The creation of the diagram is a creative abduction. Experimenting with a diagram is an attempt to track the consequences of new relations stated in the diagram. The diagram is like the premises laid down in a deductive syllogism; and the experimenting is like the inferential conclusions the reasoner tries to imply. There is also creativity in experimenting and, therefore, there is also creativity in deduction. Syllogistics has accustomed us to think that from a couple of premises follow just one determined conclusion; but in fact, several conclusions can be inferred. Not only in empirical sciences is required to imagine new experiments, but also in mathematics is required to envision new manipulations, namely questions, to make over the diagram in order to make it answer new ideas that are not explicit. Finally, observing the results expected in the experimentation on the diagram and gathering them into a new general idea is the procedure of every induction. Hence, reasoning with diagrams involves the three types of inferences.

A map is a good example of experimenting upon a kind of representative vicar diagram of a state of affairs where tracing routes, for example, are immediate experimental benefits. The story might complicate a bit though, and we can have diagrams of a major complexity, as the DNA genome databases might be. In logic, Peirce's existential graphs are precisely diagrams of thought; those graphs can be manipulated, without changing the truth value of the proposition, in order to see some information that they implicitly contain; they can also be transformed, using the five rules of inference, in order to see what information they imply (Roberts, 1963; Shin, 2002; Oostra, 2009; 2011). Besides, Shin & Lemon (2011) state that formal logic concerns valid reasoning in the form of sentences, but that is far from being the whole story of the expansion of logical knowledge. One of the recent developments of philosophers, psychologists, logicians, computer scientists, and mathematicians is the awareness of the importance of multi-modal reasoning by means of non-symbolic, diagrammatic representational systems. Peirce's existential graphs are a diagrammatic system of logic that can carry that kind of labour.

Let us unwrap the latter explanation further, the experimental aspect of the diagrams seems to be paramount for the epistemological purposes of understanding mathematical inference. Turns out that if we actually can experience those objects through the diagrammatic reasoning experience we are somehow acquainted directly with the reality of the structures under experimentation, Peirce calls this "the form of a relation" and it is straightforwardly the very object under investigation.

Two essential procedures in the determination of the meaning of mathematical statements are, according to Peirce, abstraction, and generalisation. In the case of abstraction, numbers are real because they are included in a truth expression, and the transformation of them into substantives by hypostatic abstraction does not mean anything beyond that. But this is not all the story, the fixation of the mathematical abstract entities has a causal contact, but to account for this Peirce came up with Perceptual abduction, iconic observation and experimentation, and Normative habit-rules of Reasoning. (Tiercelin, 2010).

When we perceive the kind of necessity that thirdness (continuity) imposes then its reality is a matter of course, this is the way of abductive perception of generality. The iconic capacity of diagrams gives us hold in reality by ways of prediction; for in a chemical molecular structure, e.g., we are interested mainly in the form of a relation. The iconicity and its different levels render syllogism and corollarial deduction possible.

The "form of a relation" suggests a property more than an individual. His fight against nominalism can testify this: "Peirce is strongly realist about generals, and rejects the nominalist thesis that the only things that are real are particulars: there is real thirdness. Thus the idea that structures are general (rather than particulars) would be congenial to his realism." (Hookway 2010, p. 16). Thus, Peirce would have favour, in the philosophy of mathematics, a particular flexible version of what is called "Ante Rem structuralism":

> All the properties of rational [numbers] . . . follow from the general fact that they are all contained in their order in the series constructed according to the rule. . . (Peirce, 1976, *NEM* 4, p. 580).

He insists that all that is involved in the rational numbers is The form of relation of rational consequence (Peirce, 1976, *NEM* 4, p. 580). From the epistemological point of view, the mathematical proofs can be reconstructed as a process of knowledge development within a given representational system. Peirce tells us on this:

> The first things I found out were that all mathematical reasoning is diagrammatic and that all necessary reasoning is mathematical reasoning, no matter how simple it may be. By diagrammatic reasoning, I mean reasoning which constructs a diagram according to the same precept would have the same results, and expresses this in general terms. This was a discovery of no little importance, showing, as it does, that all knowledge without exception comes from observation (Peirce, 1976, *NEM* 4, pp. 47–48).

### 3. Zalamea's advice on how to philosophise over mathematics

Fernando Zalamea (2012), a mathematician who has advanced the Peircean approach to mathematics advocates for a "synthetic philosophy of contemporary mathematics" that engages directly with the practice and creative processes of modern mathematicians. He criticises the traditional philosophy of mathematics for its focus on foundational issues and its neglect of the actual "doing" of mathematics. In this article we will retrieve his proposal as a compatible development of Peirce's conception of mathematical inference: should we adopt Peirce's ideas about mathematical reasoning and his pragmatist philosophy, then we will develop a synthetic and continuous approach to mathematics as Zalamea's approach. Let us develop, then, some key points of Zalamea's approach:

1. Engagement with Contemporary mathematics: Zalamea believes the philosophy of mathematics should be grounded in a deep understanding of current mathematical practices and developments. He

argues that many philosophers remain focused on issues and figures from the early 20th century or earlier, neglecting the significant transformations and expansions in the field since then. For Zalamea, "Mainstream philosophy of mathematics seems to have given up on the possibility of a serious dialogue with contemporary mathematical creation." (Zalamea, 2012, p. 17)

2 Emphasis on the Creative Process: Zalamea emphasises the importance of understanding the creative gestures, intuitions, and problem-solving strategies that mathematicians employ. He draws inspiration from figures like Gilles Châtelet and Albert Lautman, who sought to capture the dynamic and open-ended nature of mathematical thought. According to him, "The synthetic philosophy of contemporary mathematics proposes a new epistemology that takes the creative gestures of the mathematician as its starting point." (Zalamea 2012, p. 21)

3 Rejection of Foundationalism: Zalamea rejects the foundationalist project of reducing mathematics to a single, absolute foundation (for example, to a particular branch of mathematical thought such as set theory or logic). Instead, he sees mathematics as a pluralistic and evolving field with multiple interconnected branches and perspectives. On this, he explains: "The synthetic philosophy of contemporary mathematics rejects the search for an absolute foundation for mathematics. It recognizes the plurality of mathematical practices and the open-ended nature of mathematical creation." (Zalamea, 2012, p. 23)

4 Use of Philosophical Tools: Zalamea does not propose a completely new philosophical system. He draws on the works of philosophers like Peirce, Lautman, and Merleau-Ponty, adapting their ideas to the context of contemporary mathematics. He is particularly interested in Peirce's concept of abduction and its role in mathematical discovery. On this he says: "The synthetic philosophy of contemporary mathematics uses the tools of philosophy to understand the creative process of mathematics. It draws on the work of philosophers like Peirce, Lautman, and Merleau-Ponty to develop a new epistemology of mathematical creation." (Zalamea, 2012, p. 24)

In brief, Zalamea's vision for the philosophy of mathematics is closely intertwined with the practice of mathematics itself. It seeks to understand and illuminate the creative processes and conceptual structures that drive mathematical progress, rather than focusing solely on foundational or logical concerns. Our Peircean Abductive proposal is meant to be contrasted to contemporary cases of creativity in mathematics, in that way we build upon Zalamea's proposal.

## 4. Mathematical Creativity through Abduction: Illustrated

Peirce's abductive reasoning beautifully captures the essence of mathematical creativity – the ability to propose new ideas and hypotheses to explain unexpected observations. A hallmark of some great contemporary mathematicians lies in their impulses of creativity as the origins of their intellectual breakthroughs. Here are a few illustrative examples from contemporary mathematics:

*1. Andrew Wiles and Fermat's Last Theorem: Abduction as the logic of connecting surprising facts.*

The proof of Fermat's Last Theorem, a problem that had tantalised mathematicians for centuries, exemplifies abductive reasoning in action. Andrew Wiles's proof of Fermat's Last Theorem is a truly captivating story of abductive reasoning and mathematical tenacity. Fermat's Last Theorem, a seemingly simple statement that no three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2, has baffled mathematicians for over 350 years.

Wiles's journey began in his childhood when he first encountered the theorem. He became captivated by its simplicity and the challenge it posed. Years later, as a professional mathematician, he embarked on a secret and solitary quest to solve it. His approach involved a daring and unconventional strategy: to connect Fermat's Last Theorem, a problem in number theory, to a seemingly unrelated area of mathematics—the theory of elliptic curves and modular forms.

This connection was made possible by the Taniyama-Shimura conjecture, which proposed a deep relationship between these two seemingly disparate fields. Wiles realised that if he could prove a specific case of this conjecture, it would imply Fermat's Last Theorem. This was a bold and audacious hypothesis, a leap of faith that guided his research for years.

Wiles worked in secrecy, meticulously building a complex mathematical framework to bridge the gap between elliptic curves and modular forms. He employed a vast array of techniques, drawing on diverse areas of mathematics and developing new tools along the way. This process involved a form of diagrammatic reasoning, where he visualised and manipulated abstract mathematical objects in his mind, searching for hidden connections and patterns.

After years of intense effort, Wiles announced a proof in 1993. However, a flaw was discovered, forcing him to retreat and refine his arguments. He spent another year working tirelessly, finally achieving a complete and rigorous proof in 1994.

Wiles's work exemplifies abductive reasoning because he did not simply deduce consequences from existing theories. He observed a surprising connection between seemingly unrelated areas of mathematics, formulated a bold hypothesis, and then developed innovative techniques to prove it. This creative leap, driven by intuition and unwavering dedication, led to one of the most celebrated achievements in mathematical history. His proof not only solved a long-standing problem but also revealed deep and unexpected connections between different branches of mathematics.

Wiles encountered surprising connections between seemingly disparate mathematical objects – elliptic curves and modular forms. This observation, which did not fit neatly within existing frameworks, led him to abduce the Taniyama-Shimura conjecture, a bold hypothesis that, if true, would imply Fermat's Last Theorem. Wiles' creative leap bridged the gap between these two seemingly unrelated fields, leading to one of the most celebrated achievements in mathematical history, he stated: "The moment I had the link between modular forms and elliptic curves, I knew I had found the key to unlocking the proof. It was a moment of pure elation." (Wiles, 1995).

Wiles's struggle and eventual triumph in proving Fermat's Last Theorem beautifully illustrate the power of Peircean abduction and diagrammatic reasoning in mathematical creativity. Faced with the "surprising fact" that both his Iwasawa theory and Kolyvagin-Flach approaches failed individually, Wiles experienced a moment of abductive insight. He realised that the very reasons for their individual failures held the key to their combined success. This was not a logical deduction but a creative leap, a bold conjecture that connected seemingly disparate pieces of information. This abductive leap likely involved mental manipulation of complex mathematical structures, akin to diagrammatic reasoning, where Wiles visualised the interplay between the two approaches and perceived a way to combine their strengths.

This process led to the new idea of strengthening Iwasawa theory with tools from the Kolyvagin-Flach method. This creative synthesis, born from abduction and diagrammatic reasoning, allowed Wiles to derive the crucial class number formula and complete his proof. This exemplifies Peirce's notion of abduction as forming an explanatory hypothesis that introduces new ideas. The anecdote also highlights the role of visual and intuitive thinking in mathematics, as Wiles's mental manipulation of diagrams led him to a novel solution. Wiles's breakthrough showcases the mind's ability to make unexpected connections, formulate bold hypotheses, and arrive at creative solutions that expand mathematical knowledge. We think is well worthwhile to reproduce Wiles's own account of this fascinating moment of insight:

> I was sitting at my desk examining the Kolyvagin–Flach method. It wasn't that I believed I could make it work, but I thought that at least I could explain why it didn't work. Suddenly I had this incredible revelation. I realised that, the Kolyvagin–Flach method wasn't working, but it was all I needed to make my original Iwasawa theory work from three years earlier. So out of the ashes of Kolyvagin–Flach seemed to rise the true answer to the problem. It was so indescribably beautiful; it was so simple and so elegant. I couldn't understand how I'd missed it and I just stared at it in disbelief for twenty minutes. Then during the day I walked around the department, and I'd keep

> coming back to my desk looking to see if it was still there. It was still there. I couldn't contain myself, I was so excited. It was the most important moment of my working life. Nothing I ever do again will mean as much. (Wiles, quoted by Singh (1998: 186–187))

### *2. Grigori Perelman and the Poincaré Conjecture: Experimentation and abductive creativity in topological diagrams*

Perelman's proof of the Poincaré Conjecture, a fundamental problem in topology, also showcases the power of abductive reasoning. Grigori Perelman's proof of the Poincaré conjecture, a century-old problem in topology, is a remarkable testament to abductive reasoning and the power of geometric intuition. The conjecture, in essence, states that any simply connected, closed 3-dimensional manifold (a space with no holes and where any loop can be shrunk to a point) is topologically equivalent to a 3-sphere (the surface of a 4-dimensional ball).

Perelman's solution relied heavily on Richard Hamilton's theory of Ricci flow, a concept from differential geometry that describes how a geometric shape evolves over time, much like heat diffuses through a material. Imagine smoothing out a crumpled piece of paper by letting it "flow" according to its curvature; that is akin to Ricci flow (see Figure 1.).

However, applying Ricci flow to the Poincaré conjecture posed significant challenges. As the shape evolves, it can develop singularities—points where the curvature becomes infinite, like a sharp point forming on our paper. Perelman's key insight was to perform "surgery" on these singularities, carefully cutting them out and replacing them with smooth caps, allowing the flow to continue. This innovative technique, combined with deep geometric analysis, allowed him to control the flow and ultimately prove that any initial shape satisfying the conjecture's conditions would eventually flow towards a 3-sphere.

This solution exemplifies abductive reasoning because Perelman did not simply deduce the result from existing theorems. He observed the behaviour of Ricci flow, identified the obstacle of singularities, and then creatively introduced the concept of surgery to overcome this challenge. This involved a leap of intuition, a bold hypothesis that ultimately led to a profound and elegant solution.
Moreover, Perelman's work highlights the role of diagrammatic reasoning in mathematics. While his proof is highly technical, it relies on visualising and manipulating geometric shapes in his mind, imagining how they evolve under Ricci flow and how surgery can be used to reshape them. This intuitive, almost tactile, approach to geometry allowed him to grasp the underlying structure of the problem and develop the innovative techniques needed to solve it.

He utilised Ricci flow, a technique from differential geometry, in a novel and unexpected way. This creative application of an existing tool allowed him to make significant progress towards the conjecture, eventually leading to its complete resolution. Perelman stated: "The proof of the Poincaré Conjecture is not just about solving a single problem, but about understanding the deep structure of three-dimensional spaces."(Perelman, 2002).

Let us delve into Ricci flow again, this time highlighting how it embodies the spirit of abductive reasoning, a type of logical inference where we propose explanations for surprising observations. Imagine you are a sculptor, not chiselling away at stone, but moulding a shape made of malleable clay. You notice that when you heat the clay, its bumps and wrinkles tend to smooth out over time. This observation sparks your curiosity: Could there be a mathematical way to describe this "smoothing" process?

This is where Ricci flow comes in. It is a mathematical tool inspired by this observation, a way to model how the geometry of a shape evolves towards smoothness: visualising the shapes is a diagrammatic experiment that consists in capturing the essence of the heat-driven smoothing in a precise mathematical language.

The core equation of Ricci flow, $\partial g/\partial t = -2Ric(g)$, encapsulates this abductive leap. It links the change in the shape's metric (g) over time (t) to its curvature (Ric(g)). It is as though we were saying: "Hey, I've noticed these bumps flatten out, and it seems to be related to how curved the shape is at those points. Let's express this relationship mathematically."

This equation is not derived from pre-existing laws or axioms. It is a proposed explanation, a hypothesis formed by observing a phenomenon and seeking to understand its underlying principles. It is as though we could be reflectively saying: "I think this equation captures the essence of how this shape smooths out. Let's explore its consequences and see if it accurately predicts the behaviour I'm observing."

The abductive nature of Ricci flow becomes even more evident in its applications. When mathematicians used it to tackle the Poincaré conjecture, they were not simply applying a known formula. They were exploring a hypothesis, using Ricci flow as a lens to understand the intricate world of 3-dimensional shapes.

Grigori Perelman's breakthrough involved an even more profound abductive leap. He observed that Ricci flow could sometimes create singularities, points where the curvature becomes infinite, like sharp points forming on our clay sculpture. To overcome this, he creatively introduced the concept of "surgery," cutting out these singularities and replacing them with smooth caps, allowing the flow to continue. Again, this was not a deduction from established principles; it was a novel solution, a hypothesis born from careful observation and creative problem-solving.

In essence, Ricci flow exemplifies abduction because it is a mathematical tool inspired by observation, a hypothesis proposed to explain a phenomenon. Its development and application involve a continuous interplay between observation, intuition, and rigorous analysis, pushing the boundaries of mathematical understanding and revealing deep connections between geometry and topology.

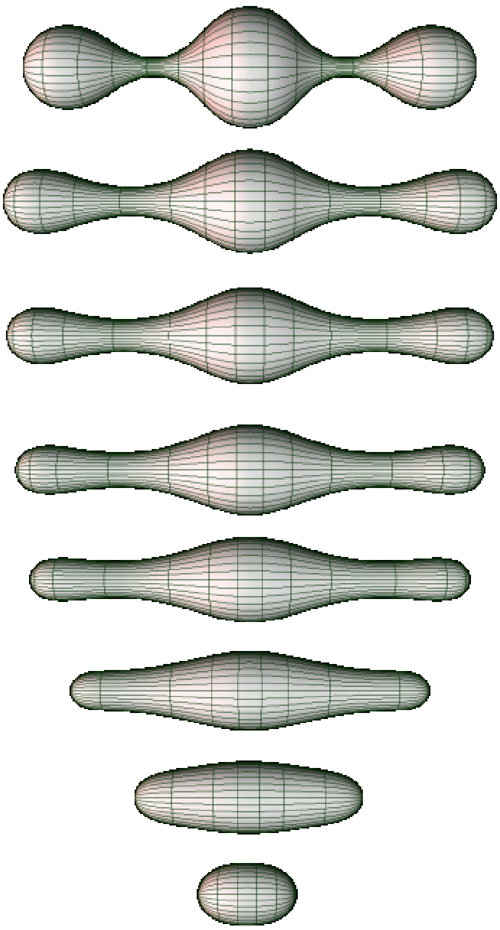

*Figure 1. Stages of the Ricci Flow. Public domain image by Wikimedia Commons.*

*3. Maryam Mirzakhani and Moduli Spaces: the Aesthetic dimension of mathematical abduction.*

Mirzakhani's groundbreaking work on moduli spaces of Riemann surfaces exemplifies the creative exploration of abstract mathematical structures. Maryam Mirzakhani made groundbreaking contributions to the study of moduli spaces, which are complex geometric objects that classify different geometric structures. Imagine them as spaces where each point represents a different shape, like a museum where every exhibit is a unique kind of curved surface.

Mirzakhani focused on moduli spaces of Riemann surfaces, which are surfaces with a particular kind of geometry (as explained above, not euclidean). She discovered unexpected connections between the geometry of these spaces and dynamical systems, the study of how systems evolve over time. This led to new insights about the "flow" of geodesics (the shortest paths) on these surfaces and their distribution within the moduli space. One of her most celebrated results involved calculating the volumes of certain moduli spaces, a problem previously considered intractable: a fundamental connection between hyperbolic geometry and complex analysis, a connection that lies at the heart of Maryam Mirzakhani's work. It beautifully exemplifies the process of abduction and the use of diagrammatic reasoning in mathematics.

She starts with a "surprising fact": the group of isometries (distance-preserving transformations) of the hyperbolic plane is the same as the group of biholomorphisms (complex analytic transformations) of the upper half-plane. This unexpected equivalence suggests a deeper connection between two seemingly distinct areas of mathematics.

This observation leads to an abductive leap: if these transformations are the same, then perhaps hyperbolic surfaces and Riemann surfaces (surfaces with a complex structure) are also deeply related. This is a hypothesis, a conjecture that seeks to explain the surprising equivalence. Mirzakhani then explored the consequences of this hypothesis. She showed that every hyperbolic surface can be given a complex structure, and conversely, every Riemann surface (with some exceptions) can be given a hyperbolic metric. This provides evidence supporting the initial abductive leap, suggesting that the connection between hyperbolic geometry and complex analysis is indeed profound.

This exploration involves a form of diagrammatic reasoning. Mathematicians visualise the hyperbolic plane and Riemann surfaces as "diagrams," mental constructs that allow them to manipulate and explore their properties. They imagine transforming these surfaces, applying isometries and biholomorphisms, and observing the results.

The equations she provided are essential tools in this diagrammatic reasoning. They provide a precise language for describing the geometry of these surfaces and the transformations that act upon them. For instance (1):

$$H = \{x + iy \mid y > 0\} \qquad (1)$$

It is an equation that defines the upper half-plane, a key "diagram" in this context.

While (2):

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2} \qquad (2)$$

Is an equation that defines the hyperbolic metric, a way of measuring distances on the hyperbolic plane. By manipulating these equations and visualising the corresponding transformations, mathematicians gain insights into the deep relationship between hyperbolic geometry and complex analysis.

Maryam Mirzakhani built upon this fundamental connection to make groundbreaking discoveries about the geometry and dynamics of moduli spaces. She used her deep understanding of

hyperbolic geometry and complex analysis to explore the "shapes of shapes," revealing hidden patterns and connections that had eluded mathematicians for decades.
This work exemplifies abductive reasoning because Mirzakhani did not start with a known theory and deduce consequences. Instead, she observed surprising patterns and connections—such as the link between moduli space volumes and the number of geodesics on a surface—and then formulated new hypotheses and techniques to explain these observations. This creative leap, bridging seemingly disparate areas of mathematics, led to a deeper understanding of the intricate structure of moduli spaces and their connection to dynamical systems. Her work, therefore, demonstrates how abductive reasoning can guide the discovery of deep relationships within the abstract world of mathematics. On her process of reasoning, Mirzakhani stated: ”The beauty of mathematics lies in its ability to connect seemingly unrelated concepts and reveal hidden patterns.” (2007, 179)

*4. Yitang Zhang and Bounded Gaps Between Primes: experimenting with bold abductive hypotheses.*

Yitang Zhang's work on bounded gaps between primes is another striking example of abductive reasoning. Zhang's breakthrough work in number theory, specifically his proof of bounded gaps between prime numbers, is a striking example of abductive reasoning in mathematics. Prime numbers, those divisible only by 1 and themselves, have captivated mathematicians for centuries. A long-standing question, the twin prime conjecture, proposes that there are infinitely many pairs of primes that differ by only 2 (like 11 and 13). While Zhang did not solve this conjecture, he made a monumental leap by proving that there are infinitely many pairs of primes that differ by no more than 70 million.

His approach involved a novel combination of existing techniques and a creative application of the "GPY sieve," a method for estimating the gaps between primes. Zhang meticulously analysed the distribution of primes, observing subtle patterns and irregularities that hinted at underlying structure. He then formulated a bold conjecture – that there exists a finite bound within which infinitely many prime pairs must fall. This was a daring hypothesis, a leap of intuition that went beyond the established knowledge.

To prove his conjecture, Zhang employed intricate and innovative methods, drawing on diverse areas of mathematics like analytic number theory and sieve theory. He meticulously constructed a complex mathematical framework, akin to a vast and intricate diagram, to capture the relationships between primes and their distribution. This diagrammatic reasoning allowed him to visualise and manipulate abstract concepts, revealing hidden connections and ultimately leading to his groundbreaking proof.

Zhang's work exemplifies abduction because he did not simply deduce consequences from existing theories. He observed unexpected patterns, formulated his bold hypothesis to explain them, and then developed new mathematical tools to prove his conjecture. He observed a pattern in the distribution of prime numbers and proposed yet a further bold conjecture: that there are infinitely many pairs of prime numbers that differ by no more than a certain fixed number. Although this conjecture remains unproven, Zhang's creative hypothesis has spurred significant research in number theory, demonstrating the power of abduction to inspire new lines of inquiry. Zhang emphasises the relevance of his inferential challenge: ”The distribution of prime numbers is one of the most fascinating and mysterious topics in mathematics. It is a puzzle that has captivated mathematicians for centuries." (Zhang, 2014:1174)

## 5. **Conclusion**

These examples illuminate how mathematical creativity thrives on abductive reasoning. It is about seeing the unexpected, the surprising connections and patterns, and then creatively proposing new ideas and hypotheses to make sense of them. As these examples show, abduction isn't just about explaining existing phenomena, it is about generating new knowledge, pushing the boundaries of mathematical understanding, and opening up exciting new avenues for exploration.

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