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Financial derivatives (based on two supports) evaluation

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Abstract
In this paper we build a PDE like Black-Scholes equation in hypothesis of a financial derivative that is dependent on two supports (usual is dependent only on one support), like am option based on gold, when national currency has a great float.
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1. Assuming the model

We suppose that two supports S, T have a generalized Brownian motion:

$$dS = A\,dt + B\,dW^1_t \quad (1)$$
$$dT = C\,dt + D\,dW^2_t \quad (2)$$

where W^1_t and W^2_t are two ρ-correlated Wiener process (named sometime as Brownian motion, Wiener-Levy process, Wiener-Bachelier process or Wiener-Einstein process, see [1], p. 123):

$$dW^1_t\, dW^2_t = \rho\, dt \quad (3)$$

and where A, B, C, D are some two-variables functions:

$$A = A(S, T, t) \quad (4)$$
$$B = B(S, T, t) \quad (5)$$
$$C = C(S, T, t) \quad (6)$$
$$D = D(S, T, t) \quad (7)$$

Remark: In main cases, A and C, respectively B and D have same formulae.

If we have a financial derivative based on these two supports. We denote with F(S,T,t) the evaluation of our derivatives at timestamp t, when current supports level are S and T. If introduce payoff function with fructification of derivative at maturity time ($t_{maturity}$), than we have a condition on boundary:

$$F(S, T, t_{maturity}) = payoff(S, T) \quad (8)$$

Other boundary conditions can be build for zero-value and infinity-value of supports:

$$F(s, 0, t) = F(0, s, 0) = 0 \quad (9)$$
$$\lim_{S\to\infty} F_S(S, T, t) = \lim_{T\to\infty} F_T(S, T, t) = 1 \quad (10)$$

2. Risk-free portofolio building and Black-Scholes type PDE

Theorem A (*extended Itō-like lemma, see [2], for Itō's lemma see [3]*): Let be S_t and T_t two stochastic precesses defined with next stochastic differential equations:

$$dS = A(S, T, t)\, dt + B(S, T, t)\, dW^1_t \quad (11)$$
$$dT = C(S, T, t)\, dt + D(S, T, t)\, dW^2_t \quad (12)$$

where W^1_t and W^2_t are two ρ-correlated Wiener process:

$$dW^1_t\, dW^2_t = \rho\, dt \qquad (13)$$

If f(S, T, t) is a differentiable function, then:

$$df = [f_t + f_S A + f_T C + ½f_{SS}B^2 + ½f_{TT}D^2 + f_{ST}\rho BD]\, dt + [f_S B]\, dW_1 + [f_T D]\, dW_2 \qquad (14)$$

Proof (see [2]).

Lemma B (risk-free portofolio): If we have a risk-free portofolio P(t), than

$$dP(t) = r\, P(t)\, dt \qquad (15)$$

where r is the annualized risk-free interest rate.

Proof. Obviously.

Theorem C (*building a risk-free mixt-portofolio with derivatives and supports*): Assuming model (see supra) the portofolio of:

1. one unit of derivative with value F(S,T,t);
2. $–F_S(S, T, t)$ units of S stock with total value $–S\, F_S(S, T, t)$;
3. $–F_T(S, T, t)$ units of T stock with total value $–T\, F_T(S, T, t)$,

Than this portofolio is a risk-free portofolio.

Proof. Let a risk-free portofolio P of:

1. one unit of derivative with value F(S,T,t);
2. a units of S stock with total value a S;
3. b units of T stock with total value b S.

Value of portofolio is:

$$P = F + a\, S + b\, T \qquad (16)$$

From (16), (14), (1) and (2) we have:

$$dP = dF + a\, dS + b\, dT = ([f_t + f_S A + f_T C + ½f_{SS}B^2 + ½f_{TT}D^2 + f_{ST}\rho BD]\, dt + [f_S B]\, dW_1 + [f_T D]\, dW_2) + a\, (A\, dt + B\, dW^1_t) + b\, (C\, dt + D\, dW^2_t) = [f_t + f_S A + f_T C + ½f_{SS}B^2 + ½f_{TT}D^2 + f_{ST}\rho BD + aA + bC]\, dt + [f_S B + aB]\, dW_1 + [f_T D + bD]\, dW_2 \qquad (17)$$

From (15), (17) and (16) we have:

$$dP = r\ P\ dt = r\ (F + a\ S + b\ T)\ dt \qquad (18)$$

From (17) and (18) we have:

$$f_t + f_S A + f_T C + \tfrac{1}{2} f_{SS} B^2 + \tfrac{1}{2} f_{TT} D^2 + f_{ST} \rho BD + aA + bC = r\ (F + a\ S + b\ T) \quad (19)$$
$$f_S B + aB = 0 \quad (20)$$
$$f_T D + bD = 0 \quad (21)$$

From (20) and (21) obtain that

$$a = -\ f_S \qquad (22)$$
$$b = -\ f_T \qquad (23)$$

q.e.d.

Corrolary D (*Black-Scholes-like PDE*) Assuming model (see supra), the value of derivative verify next PDE:

$$f_t + \tfrac{1}{2} f_{SS} B^2 + \tfrac{1}{2} f_{TT} D^2 + f_{ST} \rho BD - rF + f_S S + f_T T = 0 \quad (24)$$

Proof. From (19), (22) and (23) we have:

$$0 = f_t + f_S A + f_T C + \tfrac{1}{2} f_{SS} B^2 + \tfrac{1}{2} f_{TT} D^2 + f_{ST} \rho BD + aA + bC - r\ (F + a\ S + b\ T) = f_t + f_S A + f_T C + \tfrac{1}{2} f_{SS} B^2 + \tfrac{1}{2} f_{TT} D^2 + f_{ST} \rho BD - f_S A - f_T C - r\ (F - f_S S - f_T T) = f_t + \tfrac{1}{2} f_{SS} B^2 + \tfrac{1}{2} f_{TT} D^2 + f_{ST} \rho BD - rF + f_S S + f_T T \qquad (25)$$

q.e.d

Remarks E: If $S = T$, $A = C$, $B = D$ and $\rho = 0$ we obtain from (25) generalized Black-Scholes equation:

$$F_t + \tfrac{1}{2} B^2 F_{SS} + rSF_S - rF = 0. \quad (26)$$

If $A = \mu S$ and $B = \sigma S$ we obtain from (26) Black-Scholes equation (see [4]):

$$F_t + \tfrac{1}{2}\ \sigma^2 S^2 F_{SS} + rSF_S - rF = 0. \qquad (27)$$

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