BRAND. Broad Research in Accounting, Negotiation, and Distribution
ISSN 2067-8177, Volume 3, Issue 3, 2012

A Short Description of Electromagnetism Using the Finsler Geometry

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Abstract. It is well known that a Randers metric is a deformation of a Riemannian metric $\alpha(x,y)=\sqrt{a_{ij}(x)y^i y^j}$ using a 1-form $\beta(x,y)=b_i(x)y^i$. In this paper we are replacing $\beta(x,y)$, with $\beta_2(x,y)=\sqrt{b_{ij}(x)y^i y^j}$. We obtain a new space and we are going to study some of its properties.

Key words: electromagnetism, Finsler space, Randers spaces.

1. Introduction.

The theory of the electromagnetism is one of the most known theories of physics. Starting with Finsler’s dissertation in 1928, its study has been developed by many geometers and physicists. The Finsler geometry can indicate the behavior of the particles in an electromagnetic field.

Let M be an n- dimensional C^∞ manifold. Denote by (TM,τ,M) the tangent bundle of M . One consider the variables $x\in M$ of position and $y=\frac{dx}{dt}$ of direction and a fundamental metric function $F:TM\to R_+$ verifying the following axioms:

i) F is a differentiable function on $\tilde{TM}=TM\setminus\{0\}$ and F is continuous on the null section of the projection $\tau:TM\to M$;
ii) F is a positive function;
iii) F is positively 1-homogeneous with respect to the variables y^i ;

iv) $\forall(x,y)\in \tilde{TM}$, the Hessian of F^2 with respect to y^i is positive defined. So, the d-tensor field $g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive defined. It is called the fundamental tensor field of the space $F^n=(M,F)$.

The Finsler metric F induces a vector field

(1.1) $$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

on TM, defined by

(1.2) $$G^i = \frac{1}{4} g^{il}(x,y)\left\{\left[F^2\right]_{x^k y^l}(x,y)y^k - \left[F^2\right]_{x^l}(x,y)\right\}.$$

Any vector field in the above form (1.1) with the homogenity property

(1.3) $$G^i(x,\lambda y) = \lambda^2 G^i(x,y),\ \lambda > 0$$

is called a *spray* and G^i are called the *spray coefficients*.

2. Randers spaces.

G. Randers introduced in 1941 a special fundamental function $F(x,y)=\alpha(x,y)+\beta(x,y)$, where $\alpha(x,y)=\sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric showing the gravitational field and $\beta(x,y)=b_i(x)y^i$ is a 1-form showing the electromagnetic field. This metric was called a Randers metric.

The fundamental tensor g_{ij} of the Randers metric $F=\alpha+\beta$ is expressed by

(2.1) $$g_{ij} = \frac{F}{\alpha}(a_{ij} - \overset{1}{l}_i \overset{1}{l}_j) + l_i l_j$$

where $\overset{1}{l}_i = \frac{\partial \alpha}{\partial y^i}$, $l_i = \frac{\partial F}{\partial y^i}$, $l_i = \overset{1}{l}_i + b_i$.

The functions $G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x,y)y^j y^k$ are the components of the geodesic spray $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ and $\Gamma^i_{jk}(x,y)$ are the Christoffel symbols of the metric tensor g_{ij}. By a direct calculation [12]

(2.2) $$G^i = (\gamma^i_{jk} + l^i b_{j|k}) y^j y^k + (a^{ij} - l^i b^j)(b_{j|k} - b_{k|j}) \alpha y^k ,$$

with $l^i = \dfrac{y^i}{F}$, and $b_{j|k} = \dfrac{\partial b_j}{\partial x^k} - b_s \tilde{\gamma}^s_{jk}$ is the covariant derivative with respect to the Levi-Civita connection of the Riemannian space.
We denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), s^i_j = a^{ih} s_{hj}, s_j = b_i s^i_j,$$

$e_{ij} = r_{ij} + b_i s_j + b_j s_i, e_{00} = e_{ij} y^i y^j, s_0 = s_i y^i, s^i_0 = s^i_j y^j$ and we can write

(2.3) $$G^i = \overset{1}{G}{}^i + \frac{1}{2F}(r_{kl} y^k y^l - 2\alpha b_r a^{rp} s_{pl} y^l) y^i + \alpha a^{ir} s_{rl} y^l ,$$

or, equivalently,

(2.4) $$G^i = \overset{1}{G}{}^i + \frac{e_{00}}{2F} y^i - s_0 y^i + \alpha s^i_0 ,$$

with $\overset{1}{G}{}^i$ the components of the geodesic spray of the Riemannian space.

The Cartan nonlinear connection $\overset{C}{N}$ for the Finsler space $F^n = (M, F = \alpha + \beta)$ has the coefficients

(2.5) $$\overset{C}{N}{}^i_j = \frac{\partial G^i}{\partial y^j} .$$

Definition 2.1.*The Finsler space* $F^n = (M, F = \alpha + \beta)$ *equiped with Cartan nonlinear connection* $\overset{C}{N}$ *is called a Randers space and it is denote by* $RF^n = \left(M, \alpha + \beta, \overset{C}{N}\right)$.

The nonlinear connection $\overset{C}{N}$ determines the horizontal distribution, denoted by N too, with the property $T_u \tilde{TM} = \overset{C}{N}_u \oplus V_u$, $\forall u \in \tilde{TM}$, V_u being the natural vertical distribution on the tangent manifold TM .The local adapted basis to the horizontal and vertical vector spaces N_u and V_u is given by $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right), i = 1, ..., n$, where

(2.6) $$\frac{\overset{C}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{C}{N}{}_i^k \frac{\partial}{\partial y^k}.$$

The adapted cobasis is $\left(dx^i, \delta y^i\right), i = 1,...,n$ with

(2.7) $$\delta y^i = dy^i + N_j^i dx^j$$

The following results holds [11].

Theorem 2.1. *There exists an unique* $\overset{C}{N}$ *-metrical connection* $C\Gamma\left(\overset{C}{N}\right) = \left(F_{jk}^i, C_{jk}^i\right)$ *of the Randers space* RF^n *which verifies the following axioms:*

i) $\overset{C}{\nabla}{}_k^H g_{ij} = 0$; $\overset{C}{\nabla}{}_k^V g_{ij} = 0$;

ii) $\overset{C}{T}{}_{jk}^i = 0$; $\overset{C}{S}{}_{jk}^i = 0$.

The connection $C\Gamma\left(\overset{C}{N}\right)$ *has the coefficients expressed by the generalized Christoffel symbols:*

(2.8) $$\begin{cases} \overset{C}{F}{}_{jk}^i = \dfrac{1}{2} g^{is}\left(\dfrac{\overset{C}{\delta} g_{sj}}{\delta x^k} + \dfrac{\overset{C}{\delta} g_{sk}}{\delta x^j} - \dfrac{\overset{C}{\delta} g_{jk}}{\delta x^s}\right) \\ \overset{C}{C}{}_{jk}^i = \dfrac{1}{2} g^{is}\left(\dfrac{\partial g_{sj}}{\partial y^k} + \dfrac{\partial g_{sk}}{\partial y^j} - \dfrac{\partial g_{jk}}{\partial y^s}\right). \end{cases}$$

where $\frac{\delta}{\delta x^i}$ *are given by (2.6).*

The h- and v-deflection tensors D_j^i and d_j^i are given by

(2.8) $$D_j^i = \nabla_j^H y^i = 0$$

and

(2.9) $$d_j^i = \nabla_j^V y^i = \delta_j^i.$$

Since the h-deflection tensor D^i_j of the metrical canonical connection vanishes, in this space do not exists the interior electromagnetic field.

2. A 2nd –order Randers space.

Let $\left(M,\alpha^2\right)$ and $\left(M,\beta_2^2\right)$ be two Riemannian spaces with $\alpha(x,y)=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta_2(x,y)=\sqrt{b_{ij}(x)y^iy^j}$, $a_{ij}(x)\neq b_{ij}(x)$. Here, both quadratic forms α^2 and β_2^2 are assumed to be different and positive defined. We define a new type of Finsler metric function

(3.1) $$F_2(x,y)=\alpha(x,y)+\beta_2(x,y)$$

The function $F_2(x,y)$ has an analogy with the metric of the Randers space, but it is essentially different. The indicatrix of (3.1) is a 4th-order surface :

(3.2) $$\left[\left(a_{ij}-b_{ij}\right)y^iy^j\right]^2-2\left(a_{ij}+b_{ij}\right)y^iy^j+1=0\,.$$

We denote

(3.3) $$\overset{1}{l}_i=\frac{\partial\alpha}{\partial y^i}\ ,\ \overset{2}{l}_i=\frac{\partial\beta_2}{\partial y^i}\ \text{ and }\ \overset{\circ}{l}_i=\frac{\partial F}{\partial y^i}=\overset{1}{l}_i+\overset{2}{l}_i\ .$$

The fundamental tensor $\overset{\circ}{g}_{ij}$ of F_2 is

(3.4) $$\overset{\circ}{g}_{ij}=\overset{\circ}{l}_i\overset{\circ}{l}_j+\frac{F_2}{\alpha}\left(a_{ij}-\overset{1}{l}_i\overset{1}{l}_j\right)+\frac{F_2}{\beta_2}\left(b_{ij}-\overset{2}{l}_i\overset{2}{l}_j\right).$$

We denote

(3.5) $$\overset{\circ}{h}_{ij}=\overset{\circ}{g}_{ij}-\overset{\circ}{l}_i\overset{\circ}{l}_j$$

the angular metric of the 2nd-order metric F_2 and we get

(3.6) $$\overset{\circ}{h}_{ij}=\frac{F_2}{\alpha}\overset{1}{h}_{ij}+\frac{F_2}{\beta_2}\overset{2}{h}_{ij}\ .$$

Here, $\overset{1}{h}_{ij}=a_{ij}-\overset{1}{l}_i\overset{1}{l}_j$ and $\overset{2}{h}_{ij}=b_{ij}-\overset{2}{l}_i\overset{2}{l}_j$ are the angular metrics of the Riemannian spaces $\left(M,\alpha^2\right)$ and $\left(M,\beta_2^2\right)$.

Let us consider the Finsler function F_2 and $c:t\in[0,1]\to\left(x^i(t)\right)\in U\subset M$, a smooth curve with the property $\operatorname{Im}c\subset U$,

U being a domain of a local chart of the manifold M. We can consider the integral of action of the Lagrangian $L(x,y)=F_2^2(x,y)$ defined by

(3.7) $$I(c)=\int_0^1 F_2^2\left(x,\frac{dx}{dt}\right)dt=\int_0^1\left[\alpha\left(x,\frac{dx}{dt}\right)+\beta_2\left(x,\frac{dx}{dt}\right)\right]^2 dt .$$

The Euler-Lagrange equations are

(3.8) $$\frac{\partial(\alpha+\beta_2)^2}{\partial x^i}-\frac{d}{dt}\frac{\partial(\alpha+\beta_2)^2}{\partial y^i}=0,\ y^i=\frac{dx^i}{dt},$$

or, equivalently,

(3.9) $$\frac{d^2x^i}{dt^2}+\overset{\circ}{\Gamma}{}^i_{jk}(x,y)\frac{dx^j}{dt}\frac{dx^k}{dt}=0,\ y^i=\frac{dx^i}{dt},$$

where $\overset{\circ}{\Gamma}{}^i_{jk}(x,y)$ are the Christoffel symbols of the fundamental tensor $\overset{\circ}{g}_{ij}$. The Cartan nonlinear connection $\overset{\circ}{N}$ has the coefficients $\overset{\circ}{N}{}^i_j=\frac{1}{2}\frac{\partial}{\partial y^i}\left(\overset{\circ}{\Gamma}{}^i_{jk}\,y^j y^k\right)$.

Definition 3.1. The Finsler space $F_2^n=(M,F_2(x,y))=(M,\alpha(x,y)+\beta_2(x,y))$ equipped with the Cartan nonlinear connection $\overset{\circ}{N}$ is called a 2nd-order Randers space.

This space is non C-reducible and the particular case $a_{ij}(x)=b_{ij}(x)$ corresponds to the Riemannian space.

The nonlinear connection $\overset{\circ}{N}$ determines the horizontal distribution, denoted by $\overset{\circ}{N}$ too, with the property $T_u\tilde{TM}=\overset{\circ}{N}_u\oplus V_u$, $\forall u\in\tilde{TM}$, V_u being the natural vertical distribution on the tangent manifold TM. The local adapted basis to the horizontal and vertical vector spaces N_u and V_u is given by $\left(\frac{\overset{\circ}{\delta}}{\delta x^i},\frac{\partial}{\partial y^i}\right), i=1,...,n$, where

(3.10) $$\frac{\overset{\circ}{\delta}}{\delta x^i}=\frac{\partial}{\partial x^i}-\overset{\circ}{N}{}^k_i\frac{\partial}{\partial y^k}$$

The adapted cobasis is $\left(dx^i, \delta y^i\right), i = 1,...,n$ with

$$\overset{\circ}{\delta} y^i = dy^i + \overset{\circ}{N}{}^i_j \, dx^j \tag{3.11}$$

The following results holds [11]:
Theorem 3.1. There exists an unique N *-metrical connection* $C\Gamma\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{F}{}^i_{jk}, \overset{\circ}{C}{}^i_{jk}\right)$ *of the Finsler space* F_2^n *which verifies the following axioms:*

i) $\overset{\circ}{\nabla}{}^H_k \, g_{ij} = 0$ *;* $\overset{\circ}{\nabla}{}^V_k \, g_{ij} = 0$ *;*

ii) $\overset{\circ}{T}{}^i_{jk} = 0$ *;* $\overset{\circ}{S}{}^i_{jk} = 0$

The connection $C\Gamma(N)$ *has the coefficients expressed by the generalized Christoffel symbols:*

$$\begin{cases} \overset{\circ}{F}{}^i_{jk} = \frac{1}{2} g^{is}\left(\dfrac{\overset{\circ}{\delta} g_{sj}}{\delta x^k} + \dfrac{\overset{\circ}{\delta} g_{sk}}{\delta x^j} - \dfrac{\overset{\circ}{\delta} g_{jk}}{\delta x^s}\right) \\ C^i_{jk} = \frac{1}{2} g^{is}\left(\dfrac{\partial g_{sj}}{\partial y^k} + \dfrac{\partial g_{sk}}{\partial y^j} - \dfrac{\partial g_{jk}}{\partial y^s}\right), \end{cases} \tag{3.12}$$

where $\frac{\overset{\circ}{\delta}}{\delta x^i}$ *are given by (3.10).*

The h- and v-deflection tensors $\overset{\circ}{D}{}^i_j$ and $\overset{\circ}{d}{}^i_j$ are given by

$$\overset{\circ}{D}{}^i_j = \overset{\circ}{\nabla}{}^H_j \, y^i \quad \text{and} \quad \overset{\circ}{d}{}^i_j = \overset{\circ}{\nabla}{}^V_j \, y^i . \tag{3.13}$$

By a direct calculation we get:

Theorem 3.2. The interior electromagnetic tensor field of the space F_2^n *depend only of the fundamental function* $F_2(x, y) = \alpha(x, y) + \beta_2(x, y)$.

Conclusions. Using the results from the book [11] we can write the laws of conservation of the energy-momentum tensors and we can also study the gravitational field. We shall do this in a forcomming paper.

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