# About Multi-Heston SDE Discretization

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**Abstract**: in this paper we show how can estimate a financial derivative based on a support if assume for the support a Multi-Heston model.

**Keywords**: Euler Maruyama discretization method, Monte Carlo simulation, Heston model, Double-Heston model, Multi-Heston model.

## 1. Heston model for an asset and it's transformation

A new model assumed by Heston consists from two stochastic differential equations for a traded asset (see [1]):

(1) $dS_t = S_t \, sqrt(V_t) \, dW1_t$

where the volatility term is stochastic:

(2) $dV_t = K\,(\theta - V_t)\,dt + \varepsilon \, sqrt(V_t)\, dW2_t$

and K, θ, ε are positive constants, $W1_t$ and $W2_t$ are ρ-correlated:

(3) $dW1_t \, dW2_t = \rho \, dt$

An Euler-Maruyama discretization (see [2]) of (1) and (2) will be produce something like:

(4.0) Begin
(4.1) Δ = t[k+1]-t[k]
(4.2) X[k] = NormalValue()
(4.3) Z = NromalValue()
(4.4) Y[k] = ρ*X[k]+Z*sqrt(1-ρ*ρ)
(4.5) S[k+1] = S[k] + S[k]*sqrt(V[k])*X[k]*sqrt(Δ)
(4.6) V[k+1] = V[k] + K(θ-V[k])Δ + ε*sqrt(V[k])*Y[k]*sqrt(Δ)
(4.7) End

Applying Ito's lemma (see [3]), (1) will be:

(5) $d(\ln S_t) = -V(t)\,dt/2 + sqrt(V_t)\, dW1_t$

that will produce after Euler-Maruyama discretization of (5) and (2) produce something like:

(6.0) Begin
(6.1) Δ = t[k+1]-t[k]
(6.2) X[k] = NormalValue()
(6.3) Z = NormalValue()

(6.4) Y[k] = ρ*X[k]+Z*sqrt(1-ρ*ρ)
(6.5) LNS[k+1] = LNS[k] – V[k]*Δ/2 + sqrt(V[k])*X[k]*sqrt(Δ)
(6.6) V[k+1] = V[k] + K(θ-V[k])Δ + ε*sqrt(V[k])*Y[k]*sqrt(Δ)
(6.7) End

Comparing complexity of (4.5) and (6.5):

(7) C(4.5) = 4*C(*)+C(sqrt)
(8) C(6.5) = C(+)+C(-)+3*C(*)+C(/2)+C(sqrt)

If suppose that for float numbers we have:

(9) C(+)+C(-)+C(/2) << C(*)

than, if number of nodes in a discretizated time interval is less by $N_{max}$:

(10) $N_{max}$ = C(exp)/(C(*)-C(+)-C(-)-C(/2))

the modified version of Heston's discretization will be preffered.

## 2. Double-Heston model for an asset and it's discretization

In [4], Heston model is extent to a two stochastic semivolatilities like:

(11.0) Begin
(11.1) $dS_t = S_t\, sqrt(V_{1t})\, dW1_{1t} + S_t\, sqrt(V_{2t})\, dW1_{2t}$
(11.2) $dV_{1t} = K_1\, (\theta_1 - V_{1t})\, dt + \varepsilon_1\, sqrt(V_{1t})\, dW2_{1t}$
(11.3) $dV_{2t} = K_2\, (\theta_2 - V_{2t})\, dt + \varepsilon_2\, sqrt(V_{2t})\, dW2_{2t}$
(11.4) $dW1_{1t}\, dW2_{1t} = \rho_1\, dt$
(11.5) $dW1_{2t}\, dW2_{2t} = \rho_2\, dt$
(11.6) End

with same rules for $K_i$, $\theta_i$, $\varepsilon_i$, and $W1_{it}$ and $W2_{it}$ are $\rho_i$-correlated. Discretization code with Euler scheme is:

(12.0) Begin
(12.1) Δ = t[k+1]-t[k]
(12.2) X1[k] = NormalValue()
(12.3) X2[k] = NormalValue()
(12.4) Z1 = NormalValue()
(12.5) Z2 = NormalValue()
(12.6) Y1[k] = ρ1*X1[k]+Z1*sqrt(1-ρ1*ρ1)
(12.7) Y2[k] = ρ2*X2[k]+Z2*sqrt(1-ρ2*ρ2)
(12.8) S[k+1] = S[k] + S[k]*sqrt(V1[k])*X1[k]*sqrt(Δ)
+ S[k]*sqrt(V2[k])*X2[k]*sqrt(Δ)
(12.9) V1[k+1] = V1[k] + K1(θ1-V1[k])Δ + ε1*sqrt(V1[k])*Y1[k]*sqrt(Δ)
(12.10) V2[k+1] = V2[k] + K2(θ2-V2[k])Δ + ε2*sqrt(V2[k])*Y2[k]*sqrt(Δ)
(12.11) End

After applying an Ito's like lemma on (11.1) obtaining an alternative:

BRAND. Broad Research in Accounting, Negotiation, and Distribution
ISSN 2067-8177, Volume 4, Issue 2 – Special Issue on Financial Modeling and Investment Management, March, 2013

(13) $d(\ln S_t) = -(V_1(t)+V_2(t))\, dt/2 + \text{sqrt}(V_{1t})\, dW1_{1t} + \text{sqrt}(V_{2t})\, dW1_{2t}$

that will imply next discretization code:

```
(14.0)  Begin
(14.1)  Δ = t[k+1]-t[k]
(14.2)  X1[k] = NormalValue()
(14.3)  X2[k] = NormalValue()
(14.4)  Z1 = NormalValue()
(14.5)  Z2 = NormalValue()
(14.6)  Y1[k] = ρ1*X1[k]+Z1*sqrt(1-ρ1*ρ1)
(14.7)  Y2[k] = ρ2*X2[k]+Z2*sqrt(1-ρ2*ρ2)
(14.8)  LNS[k+1] = LNS[k] – (V1[k]+V2[k])*Δ/2 + sqrt(V1[k])*X1[k]*sqrt(Δ)
        + sqrt(V2[k])*X2[k]*sqrt(Δ)
(14.9)  V1[k+1] = V1[k] + K1(θ1-V1[k])Δ + ε1*sqrt(V1[k])*Y1[k]*sqrt(Δ)
(14.10) V2[k+1] = V2[k] + K2(θ2-V2[k])Δ + ε2*sqrt(V2[k])*Y2[k]*sqrt(Δ)
(14.11) End
```

### 3. Multi-Heston model for an asset and it's discretization

For a Multi-Heston model:

(15.1) $dS_t = S_t\, \text{sqrt}(V_{1t})\, dW1_{1t} + \ldots + S_t\, \text{sqrt}(V_{pt})\, dW1_{pt}$

(15.2) $dV_{it} = K_1\, (\theta_i - V_{it})\, dt + \varepsilon_i\, \text{sqrt}(V_{it})\, dW2_{it},\ i=1,p$

(15.3) $dW1_{it}\, dW2_{it} = \rho_i\, dt,\ i=1,p$

Before and after modification of (15.1) we have:

```
(16.0)  Begin
(16.1)  Δ = t[k+1]-t[k]
(16.2)  Tt := 0
(16.3)  For i:=1 to p do
(16.4)    X = NormalValue()
(16.5)    Z = NormalValue()
(16.6)    Y = ρ[i]*X+Z*sqrt(1-ρ[i]*ρ[i])
(16.7)    Tt += sqrt(V[i][k])*X
(16.8)    V[i][k+1] = V[i][k] + K[i](θ[i]-V[i][k])Δ + ε[i]*sqrt(V[i][k])*Y*sqrt(Δ)
(16.9)  EndFor
(16.10) S[k+1] = S[k] * (1 + Tt) * sqrt(Δ)
(16.11) End
```

and:

```
(17.0)  Begin
(17.1)  Δ = t[k+1]-t[k]
(17.2)  Vv := 0
(17.3)  Tt := 0
(17.4)  For i:=1 to p do
(17.5)    X = NormalValue()
(17.6)    Z = NormalValue()
(17.7)    Y = ρ[i]*X+Z*sqrt(1-ρ[i]*ρ[i])
```

(17.8) Vv += V[i][k]
(17.9) Tt += sqrt(V[i][k])*X
(17.10) V[i][k+1] = V[i][k] + K[i](θ[i]-V[i][k])Δ + ε[i]*sqrt(V[i][k])*Y*sqrt(Δ)
(17.11) EndFor
(17.12) LNS[k+1] = LNS[k] – Vv*Δ/2 + Tt*sqrt(Δ)
(17.13) End

## 4. Further works

We want to build some modifications of (17) as an modified IJK, BK, and TG scheme (see [5]) built on Millstein discretization scheme.

## References

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