Pricing in Multi-Heston Framework (I). Riccati equations

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Abstract

This article presents the ultimate in resolving a pricing framework's multi-Heston. Basically, we use the theorem Carr-Bakshi-Madan and a characteristic function method. In this first part, we integrate solutions of Riccati equations.

Keywords: Riccati ODE, Multi-Heston framework, financial derivatives, Carr-Bakshi-Madan theorem

1. Introduction

As an extension of Black-Scholes model (Black & Scholes, 1973), Steven and Heston (1993) define a new model with a stochastic volatility. This model was extent by Christoffersen, Heston and Jacobs (2009) as a model with two stochastic semi-volatilities. In our opinion, this model can be generalized as a stochastic model with q (q>0) stochastic partial-(or semi-) volatilities, see equation (1):

$$dS = \mu_i S\, dt + \sum_{j=1,q} v_j^{0,5} S\, dW_j$$
$$dv_j = \theta_j (\omega_j - v_j)\, dt + \xi_j v_j^{0,5}\, dB_j,\ j=1,q, \quad (1)$$

where:

1. ω_j is long term j-th partial-volatility, j=1,q;
2. θ_j return factor to mean of j-th partial-volatility, j=1,q;
3. ξ_j volatility of j-th volatility, j=1,q;
4. B_j and W_j are Wiener standard processes correlated (δ_{ij} is Kronecker symbol):

$$dW_j\, dB_j = \rho_j\, \delta_{ij},\ j=1,q,\ i=1,q\ ; \quad (2)$$

5. S is a stochastic process for a traded asset;
6. v_j is j-th partial-volatility, j=1,q.

This paper is based on a draft (Socaciu, 2015). All of proofs can be obtained in extended form there.

2. Riccati equations integration

Lemma R1. For next linear ODE:

$$dZ(x) / dx = A\, Z(x) + B, \quad (3)$$

where A and B are constants, solution is:

$$Z = -B A^{-1} + K \exp(Ax), \quad (4)$$

where K is an integration constant.

Proof. Multiply ODE with exp(–Ax).

Lemma R2. For Riccati ODE:

$$dZ(x) / dx = a\, Z^2(x) + b\, Z(x) + c, \quad (5)$$

where Z is nonnegative and:

$$Z(0) = 0, \quad (6)$$

with a, b and c constants, we have:

$$Z = 0{,}5\, [b \pm D]\, [E - 1]\, [1 - G\, E]^{-1}\, a^{-1}, \quad (7)$$

where:

$$D = [b^2 + 4\, a\, c]^{0,5} \quad (8),$$
$$G = -\, [b \pm D]\, [-\, b \pm D]^{-1}, \quad (9)$$
$$E = exp(- \pm D\, x). \quad (10)$$

Proof. After changing:

$$Y = (Z - z)^{-}{,}^{1} \quad (11)$$

ODE becomes:

$$-Y'\, Y^{-2} = a\, [z^2 + 2\, z\, Y^{-1} + Y^{-2}] + b\, [z + Y^{-1}] + c, \quad (12)$$

or:

$$Y' = -\, [a\, z^2 + b\, z + c]\, Y^2 - [b + 2\, a\, z]\, Y - a. \quad (13)$$

If:

$$z = 0{,}5\, [-\, b \pm D]\, a^{-1}, \quad (14)$$

then:

$$a\, z^2 + b\, z + c = 0, \quad (15)$$

and Riccati ODE becomes:

$$Y' = - \pm D\, Y - a. \quad (16)$$

and now apply Lemma R1:

$$Z(x) = 0{,}5\, [-\, b \pm D]\, a^{-1} - [\pm\, a\, D^{-1} - K\, E]^{-1}, \quad (17)$$

Because:

$$Z(0) = 0 = 0{,}5\, [-\, b \pm D]\, a^{-1} - [\pm\, a\, D^{-1} - K]^{-1}, \quad (18)$$

then:

$$Z = 0{,}5\ [b \pm D]\ [E - 1]\ [1 - G\ E]^{-1}\ a^{-1}. \quad (19)$$

Observation. The two solutions of Riccati ODE are identical.

Proof: Let:

$$E_+ = exp(-D\ t), \quad (20)$$
$$G_+ = -[b + D]\ [-b + D]^{-1}, \quad (21)$$
$$E_- = exp(+D\ t) = 1 / E_+, \quad (22)$$
$$G_- = -[b - D]\ [-b - D]^{-1}, \quad (23)$$

then solutions are:

$$Z_1 = 0{,}5\ [b + D]\ [E_+ - 1]\ [1 - G_+ E_+]^{-1}\ a^{-1}$$
$$= 0{,}5\ [b + D]\ [E_+ - 1]\ [1 + [b + D]\ [-b + D]^{-1}\ E_+]^{-1}\ a^{-1}$$
$$= 0{,}5\ [b + D]\ [E_+ - 1]\ [-b + D]\ [[-b + D] + [b + D]\ E_+]^{-1}\ a^{-1} \quad (24)$$

and:

$$Z_2 = 0{,}5\ [b - D]\ [E_- - 1]\ [1 - E_-/e]^{-1} a^{-1}$$
$$= 0{,}5\ [b - D]\ [E_+^{-1} - 1]\ [1 - [b - D]\ [-b - D]^{-1}\ E_+^{-1}]^{-1}\ a^{-1}$$
$$= 0{,}5\ [b - D]\ [1 - E_+]\ E_+^{-1}[-b - D]\ [[-b - D] - [b - D]\ E_+^{-1}]^{-1}\ a^{-1}$$
$$= 0{,}5\ [b - D]\ [1 - E_+]\ [-b - D]\ [[-b - D]\ E_+ - [b - D]^{-1}\ a^{-1}, \quad (25)$$

3. Integration of Riccati solutions

Corollary R3. Riccati equation:

$$dB / dt = 0{,}5\ \sigma^2\ B^2 - (b - i\ k\ \rho\ \sigma)\ B - 0{,}5\ k\ (k + i), \quad (26)$$

with initial condition:

$$B(0) = 0, \quad (27)$$

has solutions:

$$B(t) = S\ [1 - E]\ [1 - G\ E]^{-1}, \quad (28)$$

where:

$$S = [b - i\ k\ \rho\ \sigma - D]\ \sigma^{-2}, \quad (29)$$
$$D = [(b - i\ k\ \rho\ \sigma)^2 + \sigma^2\ k\ (k + i)]^{0,5}, \quad (30)$$
$$G = [b - i\ k\ \rho\ \sigma - D]\ [b - i\ k\ \rho\ \sigma + D]^{-1}, \quad (31)$$
$$E = exp(-D\ t). \quad (32)$$

Proof: In Lemma R2 let:

$$Z \leftarrow B, \quad (33)$$
$$x \leftarrow t, \quad (34)$$
$$b \leftarrow -[b - i\ k\ \rho\ \sigma], \quad (35)$$
$$c \leftarrow -0{,}5\ k\ (k + i), \quad (36)$$

$$a \leftarrow 0{,}5\ \sigma^2, \quad (37)$$

Lemma R4. If B is solution of Riccati equation:

$$dB / dt = 0{,}5\ \sigma^2 B^2 - (b - i\ k\ \rho\ \sigma)\ B - 0{,}5\ k\ (k + i), \quad (38)$$

with initial condition:

$$B(0) = 0, \quad (39)$$

then:

$$\int B(t)\ dt = S\ [t + (G - 1)\ G^{-1}\ D^{-1}\ \log(1 - G\ E) + K], \quad (40)$$

where K ia a constant and:

$$S = [b - i\ k\ \rho\ \sigma - D]\ \sigma^{-2}, \quad (41)$$
$$D = [(b - i\ k\ \rho\ \sigma)^2 + \sigma^2 k\ (k + i)]^{0,5}, \quad (42)$$
$$G = [b - i\ k\ \rho\ \sigma - D]\ [b - i\ k\ \rho\ \sigma + D]^{-1}, \quad (43)$$
$$E = \exp(- D\ t). \quad (44)$$

Proof. With notation from Lemma R3, will have:

$$\int B(t)\ dt = \int S\ [1 - E]\ [1 - G\ E]^{-1}\ dt = S \int [1 - E]\ [1 - G\ E]^{-1}\ dt$$
$$= S \int [(1 - G\ E) + (G - 1)\ E]\ [1 - G\ E]^{-1}\ dt = S \int [1 + [(G - 1)\ E]\ [1 - G\ E]^{-1}]\ dt$$
$$= S\ [t + (G - 1)\ G^{-1}\ D^{-1} \int G\ D\ E\ [1 - G\ E]^{-1}\ dt] = S\ [t + (G - 1)\ G^{-1}\ D^{-1}\ \log(1 - G\ E) + K]. \quad (45)$$

Corollary R5. If B_j, j=1,m, are solutions of equations:

$$dB_j / dt = 0{,}5\ \sigma_j^2\ B_j^2 - (b_j - i\ k\ \rho_j\ \sigma_j)\ B_j - 0{,}5\ k\ (k + i),\ j=1,m, \quad (46)$$

with initial conditions:

$$B_j(0) = 0,\ j=1,m, \quad (47)$$

then next ODE:

$$dA / dt = \sum_{j=1,m} b_j\ \theta_j\ B_j \quad (48)$$

with initial condition:

$$A(0) = 0 \quad (49)$$

will be:

$$A(t) = \sum_{j=1,m} b_j\ \theta_j\ [S_j\ t - 2\ \sigma_j^{-2}\ \log((1 - G_j\ E_j) / (1 - G_j))], \quad (50)$$

where:

$$D_j = ((b_j - i\,k\,\rho_j\,\sigma_j)^2 + \sigma_j^2\,k\,(k + i))^{0,5},\ j=1,m, \quad (51)$$
$$E_j = exp(-D_j\,t),\ j=1,m, \quad (52)$$
$$S_j = [b_j - i\,k\,\rho_j\,\sigma_j - D_j]\ \sigma_j^{-2},\ j=1,m, \quad (53)$$
$$G_j = [b_j - i\,k\,\rho_j\,\sigma_j - D_j]\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]^{-1},\ j=1,m. \quad (54)$$

Proof. Apply Lemma R4 and will obtain:

$$A = \int \sum_{j=1,m} b_j\,\theta_j\,B_j\,dt = \sum_{j=1,m} b_j\,\theta_j \int B_j\,dt$$
$$= \sum_{j=1,m} b_j\,\theta_j\,S_j\,[t + (G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log(1 - G_j\,E_j) + K_j] + K. \quad (55)$$

Now, from initial condition for A(0) obtain:

$$0 = \sum_{j=1,m} b_j\,\theta_j\,S_j\,[(G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log(1 - G_j) + K_j] + K, \quad (56)$$

wherefrom will obtain integration constant K as:

$$K = -\sum_{j=1,m} b_j\,\theta_j\,S_j\,[(G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log(1 - G_j) + K_j]. \quad (57)$$

Return to solution with replacing K:

$$A = \sum_{j=1,m} b_j\,\theta_j\,S_j\,[t + (G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log(1 - G_j\,E_j) + K_j]$$
$$- \sum_{j=1,m} b_j\,\theta_j\,S_j\,[(G_j - 1)\,G_j^{-1} D_j^{-1}\,log(1 - G_j) + K_j] = \sum_{j=1,m} b_j\,\theta_j\,S_j\,[t + (G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log(1 - G_j\,E_j)$$
$$+ K_j - (G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log\,(1 - G_j) - K_j]$$
$$= \sum_{j=1,m} b_j\,\theta_j\,[S_j\,t + S_j\,(G_j - 1)\,G_j^{-1}\,D_j^{-1}\,log((1 - G_j\,E_j) / (1 - G_j))], \quad (58)$$

Because:

$$S_j\,(G_j - 1)\,G_j^{-1}\,D_j^{-1} = [b_j - i\,k\,\rho_j\,\sigma_j - D_j]\ \sigma_j^{-2}\{[b_j - i\,k\,\rho_j\,\sigma_j - D_j]\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]^{-1}$$
$$- 1\}\ [b_j - i\,k\,\rho_j\,\sigma_j - D_j]^{-1}\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]\ D_j^{-1} = \sigma_j^{-2}\{[b_j - i\,k\,\rho_j\,\sigma_j - D_j]\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]^{-1}$$
$$- 1\}\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]\ D_j^{-1} = \sigma_j^{-2}\{[b_j - i\,k\,\rho_j\,\sigma_j - D_j]\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]^{-1}\ [b_j - i\,k\,\rho_j\,\sigma_j + D_j]$$
$$- [b_j - i\,k\,\rho_j\,\sigma_j + D_j]\}\ D_j^{-1} = \sigma_j^{-2}\{[b_j - i\,k\,\rho_j\,\sigma_j - D_j] - [b_j - i\,k\,\rho_j\,\sigma_j + D_j]\}\ D_j^{-1}$$
$$= \sigma_j^{-2}\,[-2\,D_j]\,D_j^{-1} = -2\,\sigma_j^{-2}, \quad (59)$$

then:

$$A = \sum_{j=1,m} b_j\,\theta_j\,[S_j\,t - 2\,\sigma_j^{-2}\,log((1 - G_j\,E_j) / (1 - G_j))]. \quad (60)$$

4. Next steps

Next step is to build characteristic function (Christoffersen, Heston & Jacobs, 2009) based on affine form of process. Identifying of constants in affine form in part of characteristic function will be an appeal at our results in 3rd paragraph. After obtaining characteristic functions, using

Carr-Bakshi-Medan theorem we can build an analytic solution for european call pricing problem in multi-Heston model.

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