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Algorithm for Financial Derivatives Evaluation in Generalized Double-Heston Model

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Abstract
This paper shows how can be estimated the value of an option if we assume the double-Heston model on a message-based architecture. For path trace simulation we will discretize continous model with an Euler division of time.
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1. BSM model, Heston model, Double-Heston model

From physical models, the following situation has reached acceptance: a financial asset interest rate follows a normal law, where the mean is the drift rate and the deviation is the volatility. This leads to a model that is currently accepted in finance:, the model of *geometric Brownian motion*. This model (known as *Black–Scholes–Merton model* in finance and financial engineering, see [1]) is a stochastic differential equation (1):

$$dS(t) = m\, S(t)\, dt + s\, S(t)\, dB(t), \qquad (1)$$

where:

a) $(S(t), t \geq 0)$ is a stochastic process for the value of stock;
b) m is a static parameter for the drift rate of return;
c) s^2 is a static parameter for the volatility of stock ($s \geq 0$);
d) $(B(t), t \geq 0)$ is a standard Wiener process.

Another model is assumed by Heston (see [2]) and it consists from two stochastic differential equations. *The Heston* model corrects some inconsistency of the Black–Scholes–Merton model, for example:
a) in reality, volatility is not a static parameter; it can be used as static value only on short periods (this value will obtain on calibration process, usual with a statistical estimator);
b) on long periods, it is possible that interest rate series did not verify a normal law.

The Heston model is described by the following coupled stochastic differential equations (2), (3):

$$dS(t) = A(S(t), v(t), t)\, dt + B(S(t), v(t), t)\, dB_1(t) \qquad (2)$$
$$dv(t) = C(S(t), v(t), t)\, dt + D(S(t), v(t), t)\, dB_2(t) \qquad (3)$$

where:

a) $(S(t), t \geq 0)$ is a stochastic process for value of stock;
b) $(v(t), t \geq 0)$ is a stochastic process for volatility of value of stock;

c) *A(S, v, t), B(S, v, t), C(S, v, t), D(S, v, t)* are three parametric algebraic functions;
d) $(B_1(t), t\geq 0)$ and $(B_2(t), t\geq 0)$ are two *r*-correlated standard Wiener processes, i.e. (4):

$$dB_1(t)\, dB_2(t) = r\, dt \qquad (4)$$

For Wiener processes, more details can be found in [3]. For the basic Heston model we have (5):

$$\left.\begin{array}{l} \text{a) } A = S(t)\, m \\ \text{b) } B = S(t)\, v(t) \\ \text{c) } C = K(\theta - v(t)) \\ \text{d) } D = \xi\, v(t) \end{array}\right\} \qquad (5)$$

where:
a) *m* is a drift of rate;
b) θ is long run average price volatility; as t tends to infinity, the expected value of *v(t)* tends to θ;
c) *K* is the rate at which *v(t)* reverts to θ;
d) ξ is the volatility of the volatility; as the name suggests, this determines the variance of *v(t)*.

Note that for $C = D = 0$ we obtain a static volatility model (Black–Scholes–Merton) (6)

$$dv(t) = 0.$$

The Double-Heston model (see [4]) is described by the following coupled stochastic differential equations (7), (8), (9):

$$dS(t) = M(S(t), v_1(t), v_2(t), t)\, dt + S_1(S(t), v_1(t), t)\, dB_1(t) + S_2(S(t), v_2(t), t)\, dB_2(t) \qquad (7)$$
$$dv_1(t) = C_1(S(t), v_1(t), t)\, dt + D_1(S(t), v_1(t), t)\, dB_3(t) \qquad (8)$$
$$dv_2(t) = C_2(S(t), v_2(t), t)\, dt + D_2(S(t), v_2(t), t)\, dB_4(t) \qquad (9)$$

where:

a) $(S(t), t\geq 0)$ is a stochastic process for value of stock;
b) $(v_1(t), t\geq 0)$ is a stochastic process for *half*-volatility of value of stock;
c) $(v_2(t), t\geq 0)$ is a stochastic process for *half*-volatility of value of stock;
d) $M(S, v_1, v_2, t)$, $S_1(S, v_1, t)$, $S_2(S, v_2, t)$, $C_1(S, v_1, t)$, $D_1(S, v_1, t)$, $C_2(S, v_2, t)$, $D_2(S, v_2, t)$ are three/four parametric algebraic functions;
e) $(B_1(t), t\geq 0)$ and $(B_3(t), t\geq 0)$ are two r_1-correlated standard Wiener processes, i.e. (10):

$$dB_1(t)\, dB_3(t) = r_1\, dt \qquad (10)$$

f) $(B_2(t), t\geq 0)$ and $(B_4(t), t\geq 0)$ are two r_2-correlated standard Wiener processes, i.e. (11):

$$dB_2(t)\, dB_4(t) = r_2\, dt \qquad (11)$$

g) $(B_1(t), t\geq 0)$ and $(B_2(t), t\geq 0)$ are two independent standard Wiener processes.

For the basic Double-Heston model we have (12):

$$\left.\begin{array}{l} \text{a) } M = S(t)\, m \\ \text{b) } S_1 = S(t)\, v_1(t) \\ \text{c) } S_2 = S(t)\, v_2(t) \\ \text{d) } C_1 = K_1\, (\theta_1 - v_1(t)) \\ \text{e) } C_2 = K_2\, (\theta_2 - v_2(t)) \\ \text{f) } D_1 = \xi_1\, v_1(t) \\ \text{g) } D_2 = \xi_2\, v_2(t) \end{array}\right\} \quad (12)$$

where:
a) m is a drift of rate;
b) θ_1 is long run average price volatility; as t tends to infinity, the expected value of $v_1(t)$ tends to θ_1;
c) θ_2 is long run average price volatility; as t tends to infinity, the expected value of $v_2(t)$ tends to θ_2;
d) K_1 is the rate at which $v_1(t)$ reverts to θ_1;
e) K_2 is the rate at which $v_2(t)$ reverts to θ_2;
f) ξ_1 is the volatility of the volatility; as the name suggests, this determines the variance of $v_1(t)$;
g) ξ_2 is the volatility of the volatility; as the name suggests, this determines the variance of $v_2(t)$.

Any financial derivative based on support with price $S(t)$ at time t, with quotation at time t and a value S of support as $V(S, v_1, v_2, t)$, where (12):

$$V : R_+ \times [0,T] \times [0,T] \times [0,T] \to R_+ \quad (12)$$

and at maturity time T will generate an generate an *payoff* (13):

$$payoff: R_+ \to R_+$$

For example, *European options CALL and PUT* has payoff functions (14):

$$\left.\begin{array}{l} payoff\,(x) = max\{0, x - E\} \\ payoff\,(x) = max\{0, E - x\} \end{array}\right\} \quad (14)$$

where E is *excercise price of option*.

2. Path trace simulation for option's pricing in generalized Double-Heston model

First, we discretize continuous dimension of time. Let us denote (15):

$$t[k] = t[0] + k\Delta,\ 0 \le k \le N \quad (15)$$

where:
a) $\Delta = (T - t[0])/N$
b) T is the maturity time of option;
c) N is a number of time units (like days, hours, minutes, etc); note that sometimes is used transaction days - in this case, discretization hasn't a constant step.

Because for a standard Wiener process *(B(t), t≥0)* we can obtain a standard normal random variable series *(X[B(t)], t≥0)* with (16):

$$dB(t) = X(dt)^{1/2} \qquad (16)$$

we can build a simulation step as (17):

$$\left.\begin{array}{l}
M \leftarrow M(S[k],\, v_1[k],\, v_2[k],\, t[k]) \\
S_1 \leftarrow S_1(S[k],\, v_1[k],\, t[k]) \\
S_2 \leftarrow S_2(S[k],\, v_2[k],\, t[k]) \\
C_1 \leftarrow C_1(S[k],\, v_1[k],\, t[k]) \\
D_1 \leftarrow D_1(S[k],\, v_1[k],\, t[k]) \\
C_2 \leftarrow C_2(S[k],\, v_2[k],\, t[k]) \\
D_2 \leftarrow D_2(S[k],\, v_2[k],\, t[k]) \\
S[k+1] \leftarrow S[k] + M\Delta + S_1 X_1 \sqrt{\Delta} + S_2 X_2 \sqrt{\Delta} \\
v_1[k+1] \leftarrow v_1[k] + C_1\,\Delta + D_1\ X_3 \sqrt{\Delta} \\
v_2[k+1] \leftarrow v_2[k] + C_2\,\Delta + D_2\ X_4 \sqrt{\Delta}
\end{array}\right\} \qquad (17)$$

where X_1 and X_3 are r_1-correlated, X_2 and X_4 are r_2-correlated. A simple method to generate two r correlated normal values is (18):

$$\left.\begin{array}{l}
X \leftarrow NormRand() \\
Z \leftarrow NormRand() \\
Y \leftarrow r\,X + \sqrt{1-r^2}\ \mathrm{Z}
\end{array}\right\} \qquad (18)$$

where *NormRand* is a function that produces independent real random numbers between 0 and 1, with normal distribution.

A complete simulation for interval $[t_0, T]$ in N steps with evaluation of payoff is function S*imulation*, described below:

$$\begin{array}{l}
FUNCTION\ Simulation() \\
\quad S \leftarrow S_0 \\
\quad v_1 \leftarrow v_{10} \\
\quad v_2 \leftarrow v_{20} \\
\quad t \leftarrow t_0 \\
\quad \Delta \leftarrow (T - t_0)\,/\,N \\
\quad FOR\ k \leftarrow 1,\ N \\
\qquad t \leftarrow t + \Delta \\
\qquad X_1 \leftarrow NormRand() \\
\qquad X_2 \leftarrow NormRand() \\
\qquad Y_3 \leftarrow NormRand() \\
\qquad Y_4 \leftarrow NormRand() \\
\qquad X_3 \leftarrow r_1\,X_1 + \sqrt{1-r_1^{\,2}}\ Y_3 \\
\qquad X_4 \leftarrow r_2\,X_2 + \sqrt{1-r_2^{\,2}}\ Y_4 \\
\qquad M \leftarrow M(S,\, v_1,\, v_2,\, t) \\
\qquad S_1 \leftarrow S_1(S,\, v_1,\, t)
\end{array}$$

$S_2 \leftarrow S_2(S, v_2, t)$
$C_1 \leftarrow C_1(S, v_1, t)$
$D_1 \leftarrow D_1(S, v_1, t)$
$C2 \leftarrow C_2(S, v_2, t)$
$D_2 \leftarrow D_2(S, v_2, t)$
$SS \leftarrow S + M\Delta + S_1 X_1 \sqrt{\Delta} + S_2 X_2 \sqrt{\Delta}$
$vv_1 \leftarrow v_1 + C_1 \Delta + D_1 X_3 \sqrt{\Delta}$
$vv_2 \leftarrow v_2 + C_2 \Delta + D_2 X_4 \sqrt{\Delta}$
$S \leftarrow SS$
$v_1 \leftarrow vv_1$
$v_2 \leftarrow vv_2$
END FOR
RETURN payoff(S)
END FUNCTION

3. Monte Carlo method for Option's Pricing in Double-Heston Model

Because for a level of acceptance α, where 0< α <1, a trust interval for $E[S(T)]$ is $[s - a, s + a]$, with (19):

$$s = [Simulation() + Simulation() + ... + Simulation()] / N \qquad (19)$$

and (20):

$$a = F(\alpha/2)\sigma/M^{1/2} \qquad (20)$$

where:
a) N is number of simulations;
b) F is the inverse function for CDF (cumulative distribution function) of standard normal distribution; it means that (21) or (22):

$$Prob(s - a < E[payoff(S(T))] < s + a) = 1 - \alpha \qquad (21)$$
$$Prob(E[payoff(S(T))] = s + O(M^{1/2})) = 1 - \alpha. \qquad (22)$$

where big–O notation is a Buchmann–Landau symbol (see [5]). Algorithm for evaluation of $E[payoff(S(T))]$ is described below, in *Serial_Simulation* function:

FUNCTION Serial_Simulation()
LOCAL x
$x \leftarrow 0$
FOR $i \leftarrow 1, M$
$x \leftarrow x + Simulation()$
ENDFOR
RETURN x/M
END FUNCTION

4. Further works

Like in [6] we will to parallelize Monte Carlo algorithm for generalized Double-Heston model. Also, we want to build a Merton-Garman like PDE for option pricing like in [1] for generalized Double-Heston model, and build some parallelization of PDE numerical solving, like in [4].

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6. References

[1] Fischer Black, Myron Scholes, The Pricing of Options and Corporate Liabilities, in Journal of Political Economy, 1973, 81 (3), pp. 637–654.
[2] Steven L. Heston, A closed–form solution for options with stochastic volatility with applications to bond and currency options, in The Review of Financial Studies, 1993, Volume 6, number 2, pp. 327–343.
[3] Daniel Revuz, Marc Yor, Continuous martingales and Brownian motion, second edition, Springer–Verlag 1994.
[4] Peter F. Christoffersen, Steven L. Heston, Kris Jacobs, The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work so Well, preprint SSRN, 20 febr. 2009, online at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=961037, last access: 7 mar. 2010.
[5] Eric W. Weisstein, Landau Symbols, in MathWorld—A Wolfram Web Resource, online at http://mathworld.wolfram.com/LandauSymbols.html, last access: 13 jun. 2009.
[6] Tiberiu Socaciu, Ilie Parpucea, Bazil Parv, Maria Parv, Parallel Algorithms for Financial Derivatives Evaluation in Generalized Heston Model, in Scientific Studies and Research Series Mathematics and Informatics, Vol. 19 (2009), No. 2, 479 – 492
[7] http://www.cncsis.ro/PNCDI%20II/Idei/2008/Exploratorie/Prop_finantare/Proiecte_propuse_spre_finantare_domeniul_13_3B.html, last access 1 august 2009.